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ON THE POSSIBLE LINE ELEMENTS FOR THE UNIVERSE

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1. *Introduction.*—As is well known, two different line elements, both compatible with general relativity, have been proposed respectively by Einstein and by de Sitter for the treatment of the large-scale behavior of the universe as a whole. Since neither of these line elements appears to account in any very direct or inevitable way for our present observational facts with regard to the distances and Doppler shifts for the extragalactic nebulae,[1] it is interesting to inquire whether these are the only possible line elements that could be proposed as applicable to cosmological behavior.

The present investigation will show that these two line elements, together with that given by the special theory of relativity, are as a matter of fact the only ones which are compatible with those assumptions as to the physical nature of the universe which it would seem most natural to make, and with the general theory of relativity. The possibility, however, of modifying the physical assumptions so as to permit a different type of line element will be mentioned. Opportunity will also be afforded by the article to make clear the range of circumstances which would agree with these different known line elements.

2. *The Conditions to Be Satisfied.*—As the first condition to be satisfied by the line element, we shall require it to be compatible for a limited region in space and time with the special theory of relativity and hence in polar coördinates to reduce to the form

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2 \qquad (1)$$

for small values of r, and at least for a limited range in time. The justification for this requirement is afforded by our experimental knowledge of the validity of the special theory of relativity for a limited space-time region, provided we neglect the effect of local gravitational fields, as we must obviously do in obtaining a line element for large-scale cosmological applications.

As the second condition for the line element we shall require the possibility of writing it in a form which is spherically symmetrical in the spatial variables, symmetrical with respect to past and future time, and static with respect to the time. These requirements, as is well known, lead necessarily to the form,

$$ds^2 = -e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta + e^{\nu}dt^2, \tag{2}$$

where λ and ν are functions of r alone. The requirement of spherical symmetry is an obvious one to impose, since otherwise the universe regarded on a large scale would have different properties in different directions. The requirement of symmetry with respect to past and future time means that the large-scale behavior of the universe is reversible, and the static form of the line element means that by and large the universe is in a steady state. These also are natural requirements to make, but they are not inevitable and we shall briefly return to them later.

The final conditions to be imposed on the line element arise from the fundamental relation given by the general theory of relativity connecting the metrical properties of space-time with the distribution of matter. This relation is given by the equation

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - {}^1/_2 G g_{\mu\nu} + \Lambda g_{\mu\nu} \tag{3}$$

where $T_{\mu\nu}$ is the material energy tensor, $G_{\mu\nu}$ the contracted Riemann-Christoffel tensor, G its associated invariant, $g_{\mu\nu}$ the fundamental metrical tensor and Λ Einstein's cosmological constant. In applying this equation to cosmological considerations, our scale of interest becomes so large compared even with astronomical objects, that we shall regard the matter in the universe as a perfect fluid, and hence can write for the material energy tensor the known expression

$$T^{\mu\nu} = (\rho_{00} + p_0)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} - g^{\mu\nu}p_0 \tag{4}$$

where ρ_{00} is the proper macroscopic density of the fluid, p_0 its proper pressure, and the quantities dx_μ/ds are macroscopic "velocities." The further conditions now to be imposed are that ρ_{00} and p_0 are to be constant throughout the universe, and that the spatial velocities are to be zero. These final requirements are also evidently natural ones to impose. The quantities ρ_{00} and p_0 are the macroscopic density and pressure as they would be measured by local observers, so that the first of the above requirements is merely the statement that the material in the universe shall be homogeneous when regarded from a large scale point of view. And the additional requirement that the material in the universe shall be stationary in the coördinates chosen is necessitated by the static nature

which has already been assigned to the universe by the form of line element assumed in equation (2).

3. *Dependence of ρ_{00} and p_0 on λ and ν.*—We may now proceed to the detailed calculation of the result of the conditions that we have imposed.

The line element given by equation (2) is of a standard form, and expressions for the quantities $G_{\mu\nu} - {}^1/_2 G g_{\mu\nu}$ corresponding to this line element have already been calculated.[2] By substituting these expressions in equation (3) we obtain as the only surviving components of the energy tensor,

$$\begin{aligned} -8\pi T_{11} &= -\nu'/r - (1 - e^{\lambda})/r^2 - e^{\lambda}\Lambda \\ -8\pi T_{22} &= -r^2 e^{-\lambda}\left\{\nu''/2 - \lambda'\nu'/4 + \nu'^2/4 + (\nu' - \lambda')/2r\right\} - r^2\Lambda \\ -8\pi T_{33} &= -8\pi T_{22}\sin^2\theta \\ -8\pi T_{44} &= e^{\nu-\lambda}\left\{-\lambda'/r + (1 - e^{\lambda})/r^2\right\} + e^{\nu}\Lambda \end{aligned} \tag{5}$$

where the accents denote differentiation with respect to r.

On the other hand, in our case the components of the energy tensor are also given by equation (4). Making use of the values of the metrical tensor given by our line element, introducing our requirement that the spatial velocities dr/ds, $d\theta/ds$, and $d\phi/ds$ are to be zero, and lowering suffixes twice, we easily find that equation (4) goes over into the expressions.

$$T_{11} = e^{\lambda}p_0, \quad T_{22} = r^2 p_0, \quad T_{33} = T_{22}\sin^2\theta, \quad T_{44} = e^{\nu}p_{00} \tag{6}$$

Combining equations (5) and (6) and noting that the second and third resulting equations are identical, we obtain

$$8\pi p_0 = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 + \Lambda \tag{7}$$

$$8\pi p_0 = e^{-\lambda}\left\{\nu''/2 - \lambda'\nu'/4 + \nu'^2/4 + (\nu' - \lambda')/2r\right\} + \Lambda \tag{8}$$

$$8\pi\rho_{00} = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 - \Lambda. \tag{9}$$

We note that there are two independent restrictions placed on the proper pressure by these equations. This results from our assumption of a perfect fluid, which brings with it the necessity for equal stresses parallel and perpendicular to the radius vector.

The second of the two pressure equations gives, however, a very complicated dependence on the metrical variables λ and ν, which makes it desirable to change the form of these equations before proceeding. To obtain the desired transformation, we subtract equation (7) from (8) and write

$$e^{-\lambda}\left\{\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right\} - e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \tag{10}$$

Multiplying through by $2/r$ and changing the order of terms, this can be rewritten in the form

$$e^{-\lambda}\left\{\frac{\nu''}{r}-\frac{\nu'}{r^2}-\frac{2}{r^3}\right\}-e^{-\lambda}\lambda'\left\{\frac{\nu'}{r}+\frac{1}{r^2}\right\}+\frac{2}{r^3}+e^{-\lambda}\left(\frac{\lambda'}{r}+\frac{\nu'}{r}\right)\frac{\nu'}{2}=0. \quad (11)$$

It will be noticed, however, that the first three terms of this new equation can be obtained by differentiating equation (7) with respect to r, while the last term contains the sum of equations (7) and (9). This permits us to rewrite (11) in the simple form

$$8\pi\frac{dp_0}{dr}+8\pi(\rho_{00}+p_0)\frac{\nu'}{2}=0. \quad (12)$$

And this makes it desirable to replace equations (7), (8) and (9) by the equivalent set

$$\rho_{00}=\frac{e^{-\lambda}}{8\pi}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{8\pi r^2}-\frac{\Lambda}{8\pi} \quad (13)$$

$$p_0=\frac{e^{-\lambda}}{8\pi}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{8\pi r^2}+\frac{\Lambda}{8\pi} \quad (14)$$

$$\frac{dp_0}{dr}=-\frac{\rho_{00}+p_0}{2}\nu', \quad (15)$$

which give in a very simple and convenient form the relation which connects the density and pressure of matter with the space-time metric.

4. *The Result of Imposing the Remaining Conditions.*—In the process of obtaining equations (13), (14) and (15), it will be noted that we have already imposed all the conditions discussed in section 2 except three. These are the requirements that the line element shall reduce for small values of r to the form given by equation (1) valid in the special theory of relativity, and that the proper density and pressure shall be constant throughout the universe. It is now easy to apply these further conditions.

An examination of equation (15) immediately shows that the requirement of uniform pressure can only be met by setting either $(\rho_{00}+p_0)$, or ν', or both, equal to zero. This leads directly to three alternative cases to which we can give separate treatment.

Case I. Let us first consider the case given by

$$\rho_{00}+p_0=0. \quad (16)$$

In accordance with equations (13) and (14) this immediately leads to

$$\lambda'=-\nu', \quad (17)$$

or since λ and ν must both become zero at $r=0$ in order for the line element to reduce in the neighborhood of the origin to the special relativity form given by equation (1), we may write at once

$$\lambda = -\nu. \tag{18}$$

On the other hand applying the remaining condition that ρ_{00} is a constant, we can immediately integrate equation (13) and easily obtain, as can be verified by differentiation,

$$e^{-\lambda} = 1 - \frac{\Lambda + 8\pi\rho_{00}}{3} r^2 + \frac{A}{r}, \tag{19}$$

where A is a constant of integration, and this constant must evidently be equal to zero if the line element is to reduce to the special relativity form for $r = 0$.

Hence writing for convenience

$$\frac{\Lambda + 8\pi\rho_{00}}{3} = \frac{1}{R^2}, \tag{20}$$

we obtain for the line element from equations (2), (18) and (19) the expression

$$ds^2 = -\frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + (1 - r^2/R^2)dt^2, \tag{21}$$

which will be immediately recognized as one of the known forms for the de Sitter line element.[3]

Case II. Let us now turn to the second possible case which is given by

$$\nu' = 0. \tag{22}$$

Integrating and remembering that the line element must reduce to the form (1) for small values of r we at once obtain

$$\nu = \text{const.} = 0. \tag{23}$$

On the other hand substituting (22) in (14) and solving we at once obtain

$$e^{-\lambda} = 1 - (\Lambda - 8\pi\rho_0) r^2. \tag{24}$$

Hence writing for convenience

$$\Lambda - 8\pi p_0 = \frac{1}{R^2}, \tag{25}$$

we obtain for the line element from equations (2), (23) and (24)

$$ds^2 = -\frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + dt^2, \tag{26}$$

which will be immediately recognized as one of the known forms for the Einstein line element.[4]

Case III. Finally, let us consider the case in which we have both

$$\rho_{00} + p_0 = 0 \qquad \text{and} \qquad \nu' = 0. \tag{27}$$

Under these circumstances we shall have equation (18) from Case I and equation (23) from Case II both true, which gives at once

$$\lambda = \nu = 0. \tag{28}$$

And the line element reduces completely to the special relativity form

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2, \tag{29}$$

which is also a special case of the other two line elements obtained by taking $R = \infty$.

We thus see, as stated at the beginning of the article, that the only cosmological line elements which are compatible with the conditions that were imposed, are those of de Sitter, of Einstein, and of the special theory of relativity. Let us now investigate somewhat further the relation of these three line elements to the quantities ρ_{00}, p_0, and Λ.

5. *The de Sitter Line Element.*—To obtain the de Sitter line element we had to place the sum, $\rho_{00} + p_0$, equal to zero. The proper density ρ_{00} is, however, from its nature necessarily either zero or a positive quantity. Hence, unless we are willing to assume that the idealized fluid which fills the universe is endowed with sufficient cohesion to support a relatively enormous negative pressure, we can obtain the above result only by setting density ρ_{00} and pressure p_0 each individually equal to zero. We are thus led to the conclusion that the de Sitter universe would be a substantially empty one, any matter actually in it merely producing a local disturbance away from the general form of line element.

Introducing the condition $\rho_{00} = 0$ into equation (20), we now obtain the usual de Sitter expression

$$\Lambda = \frac{3}{R^2} \tag{30}$$

connecting the cosmological constant Λ with the "radius" of the universe R.

Since the field equations (3) provide a correct description of planetary orbits, without the introduction of the cosmological term, the quantity Λ must be small enough so that the term has no appreciable effect until we consider distances greater than the radii of these orbits. Furthermore Λ must be a quantity which is independent of the coördinates, since the divergence of the left-hand side of (3) is taken as equal to zero because of the known facts as to the behavior of energy and momentum, and the divergence of the right side is then to be taken as vanishing identically. There seems to be no *a priori* reason in the case of the de Sitter universe,

however, which would make Λ necessarily a positive quantity. If Λ is positive, R is real and we have the usual closed de Sitter universe. If Λ were negative, on the other hand, R would be imaginary and the universe would not be closed.

6. *The Einstein Line Element.*—Turning now to the Einstein line element, and making use of equations (13), (24) and (25), we easily obtain the following convenient expressions for pressure and density

$$8\pi p_0 = \Lambda - \frac{1}{R^2} \tag{31}$$

$$8\pi \rho_{00} = -\Lambda + \frac{3}{R^2} \tag{32}$$

and by intercombination of these equations we can also write,

$$4\pi(\rho_{00} + p_0) = \frac{1}{R^2} \tag{33}$$

$$4\pi(\rho_{00} + 3p_0) = \Lambda. \tag{34}$$

Einstein's original deduction of his line element was based on the idea that the universe could be regarded on a large scale as filled with stars having negligible relative velocities. Hence to obtain the original Einstein result we may now put the pressure p_0 of our idealized fluid equal to zero, and write equations (33) and (34) in the well-known form

$$\Lambda = \frac{1}{R^2} = 4\pi\rho_0. \tag{35}$$

There appears to be no *a priori* reason, however, compelling us to restrict our considerations to a universe filled with material having a negligible pressure. Indeed, if the density of radiation in the universe should even be of the same order as the density of ordinary matter, the neglect of the pressure would not be permissible. For more general considerations, in which the pressure cannot be neglected, equations (31) and (32) should be used in their complete form, as has already been done by the present writer in investigating the final state of thermodynamic equilibrium in the Einstein universe.[5]

As a special case of possible importance,[6] it is interesting to consider an Einstein universe filled primarily with radiation alone, and containing, if any, only a negligible density of ordinary matter. In accordance with the relation between radiation pressure and energy density we then have

$$\rho_{00} = 3p_0 \tag{36}$$

and equations (33) and (34) become

$$\Lambda = \frac{3}{2R^2} = 8\pi\rho_{00}. \tag{37}$$

In the Einstein universe the cosmological constant Λ must in any case be regarded as a sort of parameter connected with the density or density and pressure of the universe as a whole. Its value, of course, must be small enough so that the older calculations of planetary orbits will not be affected by the introduction of the cosmological term into equation (3), and it must be independent of the coördinates in order that the divergence of the right hand side of equation (3) shall vanish identically. In accordance with equations (33) and (34), Λ and R^2 are both of them quantities which can physically hardly be other than positive. Hence the radius R of the Einstein universe is necessarily real and the universe is closed. Finally the possible physical interpretation of Λ as given by Einstein's later work[7] should be noted.

7. *The Special Relativity Line Element.*—In the case of the special relativity line element we had λ and ν both equal to zero in accordance with equation (28), which substituted in equations (13) and (14) gives us

$$\Lambda = 8\pi p_0 = -8\pi\rho_{00}. \tag{38}$$

Physically, however, this condition can be reasonably met only by taking all three quantities equal to zero. Hence the special relativity line element could be the cosmological line element only in the case of a completely empty universe with the cosmological constant Λ equal to zero.

8. *Possibility of Other Line Elements.*—In conclusion the possibility must be emphasized of obtaining cosmological line elements other than the above three, by imposing conditions different from those laid down in section 2. In particular it should be noted that our assumption of a static line element takes no explicit recognition of any universal evolutionary process which may be going on. The investigation of non-static line elements would be very interesting.

[1] See Tolman, R. C., an article to be published soon in *The Astrophysical Journal*, entitled "On the Astronomical Implications of the de Sitter Line Element for the Universe."

[2] See Eddington, A. S., *The Mathematical Theory of Relativity*, Cambridge, 1923, Equation (46.9).

[3] See Eddington, loc. cit., Equation (70.1).

[4] This form can be obtained for example by substituting $R \sin \chi = r$ in Eddington's equation (67.12), loc. cit.

[5] Tolman, R. C., THIS JOURNAL, 14, pp. 268, 343, 348 (1928).

[6] See Silberstein, L., Program for the New York meeting of the American Physical Society, February, 1929, Abstract 10.

[7] Einstein, A., *Ber. Berl.*, p. 349 (1919).

THE EXISTENCE OF RADIOACTIVE RECOIL IONS OF HIGH MOBILITY

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Introduction.—Using an air blast method of mobility measurement, Erikson[1,2] in 1924 determined the mobility of the positive ions formed about recoil atoms from radium, thorium and actinium emanations. He observed positive ions having closely the mobility of 1.56 cm./sec. per volt/cm. originally found by Franck,[3] and Franck and Meitner,[4] for the same type of ions. In addition he observed a second group of positive ions having a mobility of some 4.35 cm./sec. per volt/cm. The peaks in the mobility curves characterizing both classes of ions were, however, poorly defined in comparison to the peaks which Erikson usually[5] obtained by this method, and the ions failed to show the aging effect usually observed. Erikson first attributed the high mobilities to unclustered ions,[1] but later work convinced him that the explanation must be sought for elsewhere. He tentatively assumed the faster ion to be one having a double charge.[2] In a theoretical discussion of these results one of the writers[6,7] showed that to give such a mobility as 4.35 cm./sec. at least a triple charge or possibly even a quadruple charge was required on the ion. It was further shown that although the recoil atoms studied by various workers had always appeared singly charged[7] one might account for such multiply charged carriers of high mobility by the Auger effect together with certain theoretical deductions of Franck.[7] This explanation, however, appeared to be somewhat too artificial and it was deemed essential to verify if possible the existence of the high mobility radioactive recoil ions by a different method before further speculation was made.

Experimental Method.—In making such a test it was decided to use the Rutherford[8] alternating current method with high fields and a high frequency square-wave potential from a commutator. A commutator set capable of giving from 2000 alternations per second to 10 alternations per second was used. The method was exceedingly simple. Two circular brass plates of 10 cm. diameter, forming a parallel plate condenser were separated 0.5 cm. by means of an ebonite ring. In their centers were two circular holes 3 cm. in diameter with shoulders into which could be placed discs of 3 cm. diameter which were to serve for the source and detection of the ions. The discs were provided with insulating handles and were secured in place by spring clips. One of these plates N was faced by a nickel plate 3 mm. thick on the face which went into the chamber. The other plate B was merely brass. The one condenser plate B was grounded,

the other N was connected to the commutator brush, through a two-way switch which permitted grounding the plate or placing the potential from the commutator on it. The potential was taken from a bank of small lead accumulators whose center was grounded and whose two end terminals were connected to the slip-ring leads of the commutator. The radioactive source for these measurements consisted of a deposit of Th C obtained by rotating the nickel disc in a hot HCl solution of Radio Thorium, in equilibrium with 3 mg. of Mesothorium II, for 3 minutes. Fifteen minutes after removal from the solution this disc is coated with a deposit consisting of largely Th C. The Thorium C has a half life of 60 minutes and decays through α particle emission to Th C′. The recoiling Th C′ atoms that left the nickel disc N, could be collected on the brass disc B of the condenser under the appropriate conditions of field strength and frequency. Th C′ has a half-life period of 3.5 minutes, emitting an intense β radiation which is easily measured. These substances are therefore ideal for such measurements in that they decay sufficiently rapidly so as to make serious contamination practically negligible. The short life period of Ra C′, however, required that the electroscopic measurements be made by an observation of the *distance* fallen through by the gold leaf in a measured interval of time and taken a measured interval of time after the test plate B with Th C′ had been removed from the condenser. The microscopic eye piece on the electroscope was provided with a transparent scale divided into 0.1-mm. intervals, there being 50 such divisions in the field of view. The rate of fall of the gold leaf was tested and found to be sensibly uniform over the working range.

The procedure of making a measurement was as follows: Fifteen minutes after the removal of the nickel source N from the solution the intensity was measured, using the β radiation from the source. Then the source was placed in the lower plate of the condenser, and the collector plate B, carefully polished with the finest grade of emery paper, was inserted into the top plate. The alternating potential was then applied to the lower plate N, the upper plate B being grounded. The accelerating potential on the positive ions was always 10 or 20 volts below the negative or retarding potential in order to prevent the ions "working" across the plates over several cycles due to diffusion. The alternating potential was applied for 3 minutes and the upper plate B was removed. Sixty seconds after the removal of the plate B the measurement of its activity was begun. In this minute the electroscope was charged and brought to the zero of the scale. The scale reading on the electroscope after one minute gave the activity of the plate. The plate B was then cleaned with emery, wiped with filter paper and the process repeated using other values of the alternating potential. In this way some 18 or 20 points for various values of the potential could be made before the source N had decayed to values

which precluded further work. At the end of the 20 readings the β-ray activity of the source N was again taken and the two values were checked

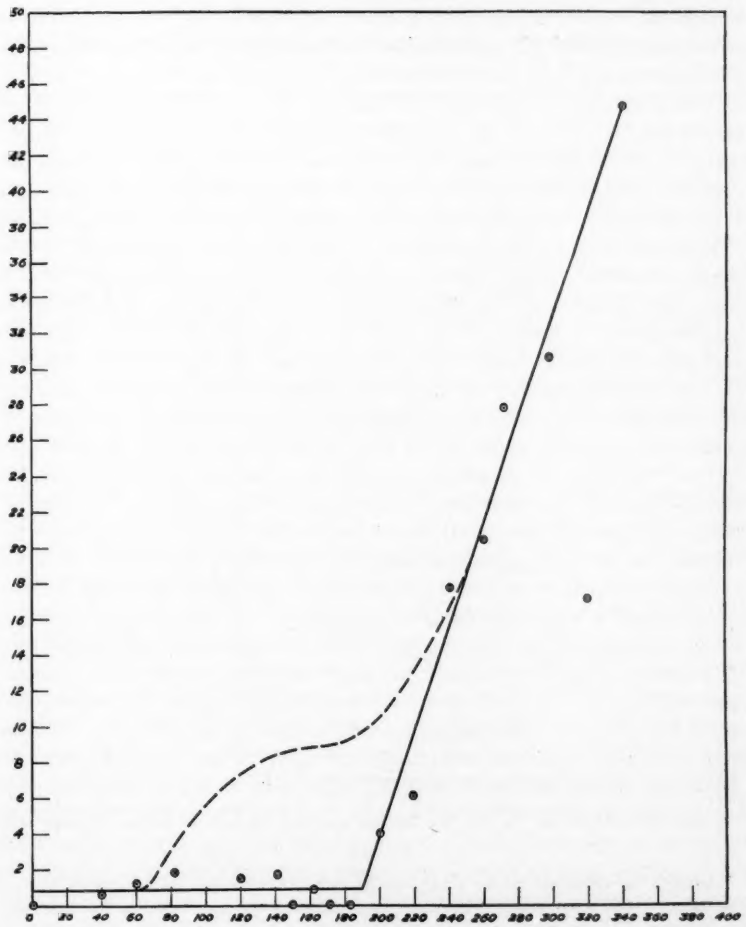

FIGURE 1

Currents in arbitrary units, abscissae, against potential in volts between plates, ordinates. 910 alterations per second, plate distance 0.5 cm., mobility 1.2 cm./sec. per volt/cm.

against the standard decay curve for the Ra C source. The measurement were made both with increasing and decreasing potentials. The curves between the potential and the activity of B were plotted after correcting

the activity for the decay of the source. The gas used was obviously air from the room at 75 to 76 cm. pressure and 20°C.

Results.—The present investigation was undertaken for the purpose of looking for the high mobility positive ion of 4.35 cm./sec. per volt/cm. value. In Erikson's[1,2] experiments there appeared to be about equal

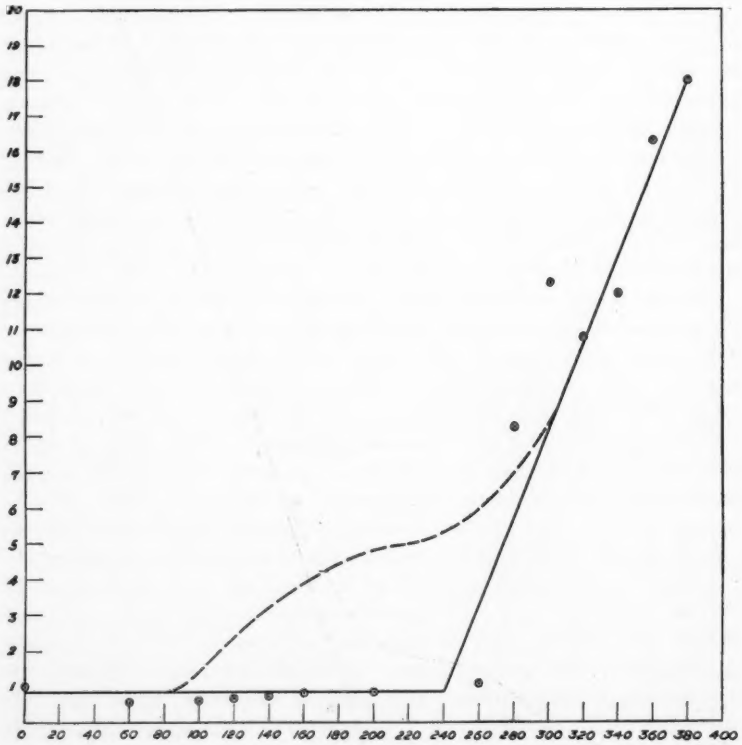

FIGURE 2

Currents in arbitrary units, abscissae, against potential in volts between plates, ordinates. 1642 alternations per second, plate distance 0.5 cm., mobility 1.85 cm./sec. per volt/cm.

numbers of this ion and the normal ion of 1.56 cm./sec. mobility. In spite of the fact that Erikson observed no aging of his fast ion it was felt essential in these experiments to work over time intervals fully as short as those of Erikson since the high mobility had not been observed by the earlier workers using long time intervals for the crossing of the ions.[6] Thus while some measurements were made using frequencies as low as

30 alternations per second most of the measurements were made at 900 or 1600 alternations. This gave a large range of potentials before the normal ions could cross in which to search for the fast ions. At 1600 alternations the fields required to drive the ions across from N to B were so high that not much of the curve with ions crossing could be observed. Therefore to gain an idea of the saturation value of the currents lower fields were resorted to. The accuracy of individual readings was limited by the method of reading the electroscope, for the gold leaf has a tendency to "jump" if observed over short time intervals and for short ranges of deflection only. These errors were never consistent and so cancelled out. Readings were not taken with the purpose of getting the value of the break points and calculating the mobility from the well-known equation applicable to this method, i.e., $k = Nd^2/V_0$, where N is the frequency, d the plate distance 5 mm. and V_0 is the intercept of the mobility curve with the axis. They were taken to cover as *wide* a range of potentials as possible over the region where the fast ions might appear, for the purpose of detecting the two classes of ions if there were any present. In figures 1 and 2 the full curves represent two of the curves obtained with this method. The other curves obtained were quite similar but the points on some were not as regular as those reproduced. Both of these curves indicate clearly that within the limits of measurement the points show practically no ions crossing at values of the potential much less than 70% of the critical potential, and those appearing to cross early are of such a nature as to have no significance. Had ions of the type found by Erikson been present to even as much as 20 per cent of the total ionization the dotted curves in figures 1 and 2 would have been obtained. In no case of the dozen or so curves taken was there ever found a curve of the type indicated by the dotted lines with a mobility in the neighborhood of 3 to 4 cm./sec. per volt/cm. What irregularities were observed before the normal ions of mobility 1.5 cm./sec. crossed were in all cases less than 5% the total number of ions present as shown by the value of the saturation curve. The rough average value of the positive-ion mobility found here was 1.51 cm./sec. per volt/cm. This value must not be taken as significant as the emphasis was placed on the parts of the curves where the fast ions were due to appear and not on break points. The time intervals for the ions to cross was in most cases from 0.0006 second to 0.0012 second though some results were obtained for as high as 0.03 second. Thus the ions were measured in air at atmospheric pressure in time intervals after formation which ranged from longer periods to shorter periods than in Erikson's experiments.

Conclusions.—It thus appears that for the positive ions of Th C′ formed from recoil atoms of Th C there was observed in air, over time intervals from 0.0006 second to 0.03 second, only the normal positive radioactive

recoil ion of mobility of some 1.5 cm./sec. per volt/cm. present to more than 5%. This result, while in agreement with the recent results of Dee[9] and the earlier observers and with what one might expect from more plausible theory, is radically different from that obtained by Erikson. Whether the source of the discrepancy lies in the use by Erikson of the *radioactive emanations* in his air blast which could have infected his apparatus in a regular way so as to give a second peak (a possibility which Erikson has doubtless carefully eliminated, though he does not tell how), or whether the differences lies in a fundamental difference in the nature of the recoil atoms of Ra emanation or Ra A and of Th C one cannot at this juncture state. One must therefore regard with caution the existence of the high mobility radioactive recoil ions of Erikson and withhold further speculation as to their origin until these shall have been confirmed by some method free from possibilities of contamination.

[1] Erikson, H. A., *Physic. Rev.*, **24**, 622, 1924.
[2] Erikson, H. A., *Ibid.*, **26**, 629, 1925.
[3] Franck, J., *Ber. Dtsch. Phys. Ges.*, **11**, 397, 1909.
[4] Franck and Meitner, *Ibid.*, **13**, 671, 1911.
[5] Erikson, H. A., *Physic. Rev.*, **20**, 117, 1922.
[6] Loeb, L. B., *Proc. Nat. Acad. Sci.*, **13**, 703, 1927.
[7] Loeb, L. B., *Science*, **67** 468, 1928.
[8] Rutherford, E., *Phil. Mag.*, **44** 433, 1903.
[9] Dee, P. I., *Proc. Roy. Soc.*, **116A**, 664, 1927.

AN INTENSITY GAUGE FOR "SUPERSONIC" RADIATION IN LIQUIDS

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The high frequency sound waves of high intensity (often called "supersonic" as distinguished from low intensity "ultrasonic" waves) recently developed by Wood and Loomis,[1] and applied by Richards and Loomis[2] to several chemical phenomena, have produced effects in liquid systems, some of which are extremely difficult to account for on the primary properties of compressional waves. Notably the increase in reaction velocities, which has also been recently observed by other investigators,[3] has at present no direct or plausible explanation. Attempts to explore further the mechanism of acceleration of reactions have met with many difficulties, as is to be expected in so undeveloped a field. Chief among these has

been the lack of a reliable measure of the intensity of the compressional waves. This communication will outline several methods of attack on this problem, stating their most patent experimental objections. It will also describe briefly the operation of the simple principle which has, to date, been found most effective.

Perhaps the most obvious method of measuring intensity depends on maintaining a weighted reflecting surface in position by sound waves. A knowledge of the extent of surface, and of the weight just insufficient for its displacement gives a direct estimate of the radiation pressure and hence, by straightforward means, leads to an evaluation of the energy density, internal temperature differences, etc., in the liquid. Wood and Loomis[1] describe an experiment in which an 8-cm. glass disc supported 150 grams in radiated oil. Several very different modifications of this principle were employed without obtaining reproducibility of more than order of magnitude in any case. Success in this method depends upon a regular standing wave system, and failure may be attributed to its absence. Not only are the large amplitude vibrations of the quartz crystal far from pure, but reflection from the walls and other secondary phenomena produce a complex and shifting interference pattern in the liquid.

Several phenomena described by Richards and Loomis[2] are amenable to quantitative treatment, but none of them are fitted to the measurement of sound intensity. The ebullition of liquids immediately below their boiling points and the cavitation of air in liquids are unsuitable both because they require too great precaution for routine measurement, and because they give no direct information concerning forces in the liquid. The emulsification of one liquid in another has, under suitable conditions, been shown[4] to be reproducible, but yields results of little value since it is due to transverse waves in glass vessels and not to compressional waves in the liquid itself.

An attempt was made to measure sound intensity by the rise in temperature of a totally absorbing body. One-half of a set of copper-constantin junctions were wound with rubber, and the other half encased in a thick glass tube. Both were then placed in the irradiated liquid and the difference in potential between them measured. The absorbing elements maintained a considerably higher temperature than the reflecting but, owing to irregular convection currents in the liquid and to lag in heat transmission through the rubber casing, the difference in temperature was not clearly indicative of small differences in sound intensity.

The heating of liquids, notably water, has already been employed[2] as a rough measure of relative sound intensity. This phenomenon is, however, extremely complex, and is not suitable at present for exact work in this connection. High frequency fields produce a dielectric loss in the liquid which manifests itself as heat, and hence scrupulous electrical shielding is

essential. Moreover, it is not at present clear how much sound is converted into heat per unit path through the liquid. If the measurement is made in glass vessels reflection phenomena and the transverse waves in glass make the conditions so complex as to be wholly incalculable. Finally, the absorption of sound by a liquid, due to the deviation from perfectly adiabatic conditions in its transmission, involves changes of temperature so small as to require a very refined technique for their detection. This matter is being made the subject of a special investigation which will appear elsewhere.

Two further principles deserve brief mention for, although unsuccessful, they are indicative of sound wave effects. The first depends upon the change in volume of small manometric bulbs. Thin bulbs with flattened sides were blown on small-bore, soft-glass capillary tubes, calibrated against known pressure differences, and subjected to sound radiation. Volume changes corresponding to pressures of 0.5 mm. mercury were noted, but these were considered unsatisfactory because, in order to indicate absolutely the radiation pressure, the surface of the bulb must be totally reflecting. Moreover, when sufficiently sensitive to register more than the order of intensity of radiation the bulbs were so fragile that they exploded under the influence of sound waves. The second principle is quite different, and depends upon the retardation of flow of a liquid in a capillary tube. The rate of flow being a function of the difference of pressure which causes it, it is possible to construct a modified viscosimeter which gives results of almost any desired sensibility. Although more satisfactory than all the methods except that to be described below, this gauge was also abandoned for two reasons: firstly, it was found possible, with sufficiently high sound intensities, to cause an upward or reverse flow in the capillary tube, and secondly, resonance phenomena in the column of liquid were found to play so important a part that, even with low sound intensities, the flow was sometimes indefinitely arrested.

The gauge finally adopted depended for its operation upon the ear-trumpet principle, the concentrated sound energy being measured as hydrostatic pressure. A small funnel was blown on the end of a fine capillary tube and immersed in the liquid radiated. The upward displacement of the capillary meniscus gave a measure of the sound-wave intensity. This device was tested with sound waves of frequencies between 250 and 500 K.C., and intensities varying from that produced by a crystal driven by a 7-watt tube, where the rise in a 0.75-mm. capillary was of the order of 0.01 cm., to that corresponding to 1 K.W., driving energy, which gave about 50 cm. rise with the same funnel. Its behavior was perfectly consistent over this range. Indeed, using a 75-watt tube, the standing waves in water were measured by raising and lowering the funnel, and a satisfactory value was obtained for the velocity of sound

in water. The maximum upward displacements were about 0.40 cm. and the minimum about 0.01 cm., these values recurring regularly as the standing wave system was explored. This measurement demonstrates two points of interest: first, that the gauge might be used, if it seemed profitable to develop it to the necessary degree of precision, to measure the coefficient of absorption of sound in a liquid; and second, that tuning effects may play a large part in the magnitude of the upward displacement of the meniscus.

Tuning effects are of two types. The first consists of specific resonance phenomena in the capillary column, and causes a jerky upward motion which takes some seconds to attain its full magnitude. This appears to cause no serious error in the ultimate reading; possibly the strong heating of liquid in the neighborhood of the capillary meniscus facilitates, by lowering the surface tension, the adjustment of the gauge. The second type of tuning is caused by variations in the position of the funnel in the standing wave system in the main body of radiated liquid, as illustrated above. Enormous differences in deflection at uniform intensity may be traced to this effect, and it is obviously important to adjust the funnel, when regular standing waves are present, to a position of maximum deflection in order to secure comparable results for different intensities. When, as in glass test tubes, the waves pattern is extremely complex because most of the energy is transmitted to the liquid by transverse waves in the glass, tuning effects are far more erratic, and only by replacing the funnel exactly in the same position in the same tube in a constant volume of water may reasonably comparable results be obtained.

Several points should be mentioned concerning the construction and operation of this instrument. If fine capillaries are used it is necessary that the liquid be wholly air-free; otherwise small bubbles liberated by cavitation at once block action in the capillary. Since water dissolves less air than most common liquids it may conveniently be used in routine work. Since it is only with a perfectly clean capillary bore that the best results are obtained, complete freedom from grease is also desirable. The diameter of the capillary may vary within considerable limits, the amount of upward displacement for a given intensity being roughly in inverse proportion to the square of the radius. The capillary must not, however, be much less than 0.5 mm. since a point is reached where resistance to flow of the liquid is too great to permit response to the small pressures involved. Tubes up to 2.5 mm. give perfectly satisfactory results, but the deflections they show are so small that maximum precision may be secured only with the use of a microscope. Variations in the properties of funnels of different configurations are considerable: a thick-walled funnel of roughly exponential form appears to be most satisfactory.

It is apparent that this principle gives only a relative and not an absolute

measure of sonic intensity. This must be the case unless reflections from the funnel wall are perfect, and unless no interference takes place during the concentration of sound. Not only are these conditions unfulfilled, but much sound is transmitted through the walls of the funnel and lost. A lower limit only may therefore be assigned to radiation pressure on the basis of this instrument. It has been demonstrated, however, that it would function reliably as an absolute gauge if means could be found to calibrate it.

Summary.—A number of methods of measuring high sonic intensities in liquids have been discussed, and the difficulties of securing absolute measurements emphasized. A gauge of the ear-trumpet type is described which is capable of giving simply reproducible values.

[1] R. W. Wood and A. L. Loomis, *Phil. Mag.*, (vii), **4**, 417 (1927).

[2] W. T. Richards and A. L. Loomis, *J. Am. Chem. Soc.*, **49**, 3086 (1927).

[3] F. O. Schmidt, C. H. Johnson, and A. R. Olson, *J. Am. Chem. Soc.*, **51**, 370 (1929).

[4] W. T. Richards, submitted to *J. Am. Chem. Soc.*, Feb. 14, 1929. To be published probably in June.

HYDROGEN-ION WAVE FUNCTION

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Burrau[1] has evaluated the energy of the positive hydrogen-molecule ion in the normal state as a function of the internuclear distance by direct numerical computation of the energy parameter E in the Schrödinger differential equation. This method gives any desired degree of approximation for the energy, but does not produce a compact analytic expression for the wave function.

In the present paper an approximate wave function for the ion is developed, and its usefulness is demonstrated by comparing its energy values with those calculated by Burrau. The method is briefly as follows: Schrödinger[2] has shown that the correct eigenfunctions of a system with the potential energy V are those which render the integral

$$J = \int \left[\frac{h^2}{8\pi^2 m} \operatorname{grad} \psi \operatorname{grad} \bar{\psi} + V \psi \bar{\psi} \right] d\tau \qquad (1)$$

an extremum, subject to the condition that ψ be normalized. By Green's theorem, J may be changed to the form

$$J = \int \psi H \bar{\psi} \, d\tau \qquad (1')$$

where H represents the operator

$$-\frac{h^2}{8\pi^2 m}\Delta + V$$

The stationary values of J are the eigenwerte E_n of the system. In particular, for the normal state ($\psi = \psi_0$), J is an absolute minimum, i.e., $J(\psi_0) = E_0$. Thus, if we assume for ψ_0 an approximate function

$$\psi(\alpha, \beta, \ldots\ldots) \qquad (2)$$

where α, β, . . . are parameters, then the best values of these parameters will be obtained by introducing (2) into (1′) and minimizing J with respect to them. Obviously, the degree to which ψ will approximate ψ_0 (and the resultant J will approximate E_0) will depend greatly upon the particular form chosen for ψ. In the present case we are guided in our choice of ψ by the following considerations:

The differential equation is separable in the elliptic coördinates λ, μ. Since the normal state is non-degenerate, this demands that ψ be of the form $\Lambda(\lambda)\,\mathrm{M}(\mu)$.

Since the differential equation is an even function of μ, $\psi(\mu) = \pm\psi(-\mu)$. The absence of nodes in the normal state requires the positive sign.

At zero internuclear distance, ψ becomes the solution of He: $\psi = \frac{1}{\sqrt{\pi}} e^{-2r}$, while for infinite separation ψ reduces to the hydrogen function: $\frac{1}{4\sqrt{\pi}}(e^{-r_a} + e^{-r_b})$.

These properties are satisfied by the function[3]

$$\psi = C(\alpha,\beta,R)e^{-1/2R\alpha\lambda}\,(e^{-1/2R\beta\mu} + e^{1/2R\beta\mu}) \qquad (2')$$

where

$$\lambda = \frac{r_a + r_b}{R}, \quad \mu = \frac{r_a - r_b}{R}$$

R being the distance from nucleus a to nucleus b and r_a, r_b the radii to the electron from these nuclei, all in units of $a_H = 0.532\ 10^{-8}$ cm. α ($\alpha_{R=0} = 2$, $\alpha_{R=\infty} = 1$) allows the charge density $\psi\bar{\psi}$ to concentrate with elliptic symmetry about the nuclei as R decreases, while β ($\beta_{R=\infty} = 1$) permits a net increase of charge density between the two nuclei.

Assuming fixed nuclei, the Hamiltonian operator in elliptic coördinates becomes:

$$H = \frac{-4}{R^2(\lambda^2 - \mu^2)}\left[\frac{\partial}{\partial\lambda}(\lambda^2 - 1)\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\mu}(1 - \mu^2)\frac{\partial}{\partial\mu} + 2R\lambda\right] + \frac{2}{R} \qquad (3)$$

Here lengths are in units of a_H and energy in units of $E_H = \frac{2\pi^2 me^4}{h^2}$.

By introducing (2′) and (3) into (1′), we obtain the following expression for J in terms of α, β and R:

$$J = \frac{f_1(\beta R) + (\alpha R - 4R - 4/\alpha + 1) f_2(\beta R) - {}^2/_3(\beta R)^2}{{}^1/_3 \beta^2 f_1(\beta R) + (R/\alpha + {}^1/\alpha^2) f_2(\beta R) + {}^2/_3} + \frac{2}{R} \quad (4)$$

where

$$f_1(\beta R) = e^{\beta R}(1 - {}^1/\beta R) + e^{-\beta R}(1 + {}^1/\beta R)$$

and

$$f_2(\beta R) = \frac{1}{\beta R}(e^{\beta R} - e^{-\beta R}) + 2.$$

In the actual computation, E (the minimum of J), for various values of R, was found by graphical interpolation of calculated values in the neighborhood of the minimum. The work was greatly facilitated by the fact that the correct values of α and β can be estimated fairly well beforehand and that the value of α which makes E a minimum with respect to α is only slightly affected by changes in β, and vice versa.†

The results are given in table 1. The last column contains the values of E calculated by Burrau, which are correct to within ± 0.002.

TABLE 1

R	α	β	E	E (BURRAU)
0.1	1.95	1.00	16.044	
1.0	1.59	0.99	−0.903	−0.896
1.3			(−1.112)*	−1.110
1.6			(−1.191)	−1.186
1.8			(−1.203)	−1.198
2.0	1.36	0.90	−1.205	−1.204
2.2			(−1.199)	−1.200
2.4			(−1.192)	−1.192
2.95			(−1.155)	−1.158
3.00	1.22	0.89	−1.154	
4.00	1.14	0.88	−1.092	
			−1.000	−1.000

* The values in parentheses have been interpolated to afford a comparison with those calculated by Burrau. The interpolation was performed on the graph of $E^* = E + 2/R$, which gives very nearly a straight line for values of R near the equilibrium value.

It is not to be expected that the eigenfunction (2′) is as nearly correct as the good agreement of the energy values might suggest, since the variation method gives better "eigenwerte" than "eigenfunctions."[4]

The energy, as a function of R, is shown graphically in figure 1.

In comparison with the above minimum value of E (-1.205)[5] we may quote the values obtained by the conventional perturbation method. The first order term gives $E_{\min}$ -1.13, while the second order term decreases this to -1.33[6,7] which is known to be too low.

We may also test the validity of the function (2′) for large values of R,

where the perturbation due to the free nucleus is sensibly that of a uniform electric field, by comparing its energy values with those of the conventional perturbation method. In the units used in this paper, the variation method gives $-4/R^4$, in comparison to $-4.5/R^4$ obtained by the perturbation method.

Classically the above polarization energy is assumed to give the chief contribution to the interaction of an atom with an approaching charged particle. At $R = 4$, where the hydrogen nucleus is still well "outside"

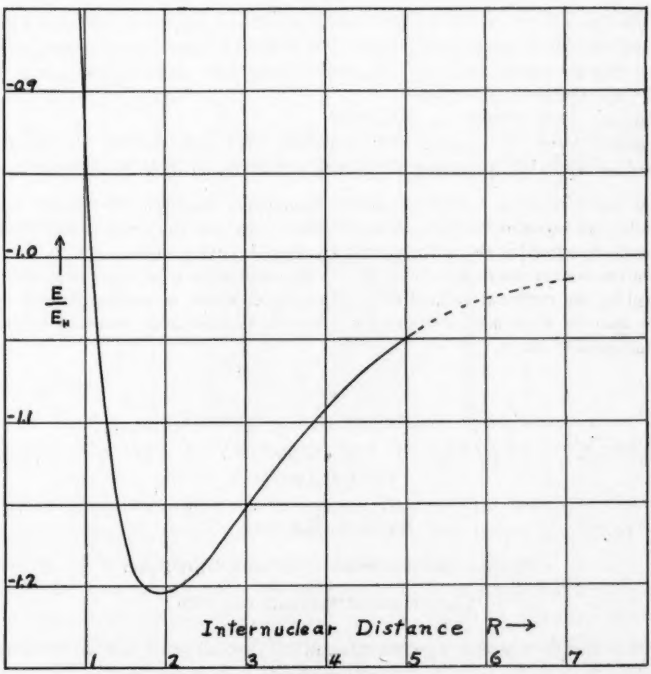

FIGURE 1

of the atom, this polarization energy is -0.018, as compared to the correct value -0.092 calculated above. This illustrates the inadequacy of the classical concepts of interatomic forces.

The authors wish to express their indebtedness to Professor Kemble for suggesting the application of the variation method of attack, and for his advice and criticism in the preparation of this paper.

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[1] O. Burrau, Kgl. Danske Vidensk. Selskab., *Math-fys Meddelelser.*, vii, 14 (1927).

[2] Schrödinger, *Ann. Physik*, **79**, 361 (1926). Schrödinger's equation (23), p. 376,

goes over into equation (1) above when T is a quadratic form in $\frac{\partial\psi}{\partial q}$.

[3] Finkelstein and Horowitz, *Zeits. Phys.*, **48**, 118 (1928), assume a similar form for the function of the ion, but with the single parameter Z (nuclear charge). This is the same as (2′) with $\alpha = \beta$.

[4] The function ψ may be tested by substitution into the differential equation $H\psi = E\psi$. As an example of the agreement obtained in this way by the function (2′) we may quote the following values of $I = (H\psi - E\psi)/E\psi_{max}$:

(λ, μ)	(1,0)	(1.5,0)	(2,0)	(2.5,0)	(1, ±.5)	(2, ±1)
I	−0.022	0.006	0.002	−0.001	−0.06	−0.006

[5] Finkelstein and Horowitz, loc. cit., obtain the value: $E_{min} = -1.166$ from a method similar to the one of the present paper. The large discrepancy is undoubtedly due to the fact that the single parameter (Z) used by them does not introduce enough "flexibility" into the wave function.

[6] Pauling, *Chem. Reviews*, **v**, 173 (1928).

[7] Unsöld's value of −1.205, from a second order perturbation calculation, was obtained by neglecting resonance terms and is therefore of little significance.

† *Note added in proof:* After one of the parameters has been determined approximately by the variation method, an approximate value for the second may be found very easily by applying the criterion that $H\psi$ should remain finite at the nuclei. The value of the second parameter found in this way will be in good agreement with that obtained by the variation method only if the eigenfunction is nearly correct. In the present case the agreement obtained for β by the two methods was well within the probable error of either.

LATENT CARRIERS OF ELECTRICITY IN THE GASEOUS DISCHARGE

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1. In course of some experiments on the discharge of electricity through gases in which large hollow electrodes of special construction were used the author has observed considerable ionization currents inside the electrodes. These currents were too large to be caused by diffusion of ions from the discharge tube, and in order to investigate the phenomenon under more favorable conditions the following apparatus was constructed. $P, P \ldots P$ (Fig. 1) is a system of parallel circular plates, 15 cm. diam. and 2.5 mm. apart, provided in the center with circular apertures 5 cm. diam. In the hollow cylinder formed by these apertures an ionization chamber is placed consisting of a metal gauze cylinder G and a coaxial electrode C. The upper plate has no aperture and in this way closes in the ionization chamber. A discharge is passed under a large bell-jar between the system of plates

serving as one electrode and another electrode R in shape of a heavy rod. At the same time a potential difference is put across each pair of plates in the system sufficiently large to absorb completely all the electrons, the high speed positive ions or other carriers of electricity which may be present in the discharge, and thus to prevent them from penetrating into the ionization chamber. It may be easily shown that for this the following relationship must be satisfied: $\frac{V}{v} < \left(\frac{l}{2d}\right)^2$, where V is the potential difference between the electrode R and the system of plates, v the p. d. across each pair of plates, l the distance between the edge and the aperture of the plates, and d the distance between the plates, independently of E/M, the ratio of the charge to the mass of the carrier of electricity. Since in the apparatus $l = 5$ cm., $d = 2.5$ mm. and V was usually less than 1000 volts, a p. d. of less than 10 v. across the plates was sufficient to prevent all the electrons and positive ions from penetrating into the

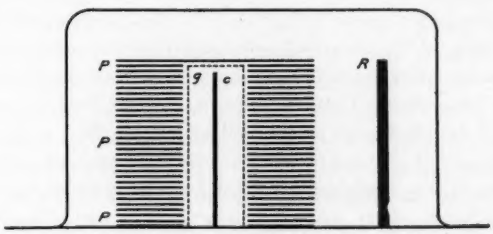

FIGURE 1

ionization chamber. For reasons given further on, however, the p. d. applied to the plates was kept much higher, just below the point where ionization by collision starts between the plates.

In the ionization chamber the electrode C is grounded through a galvanometer while the gauze cylinder G can be raised to a desired potential. Care was taken throughout the experiments to avoid ionization by collision between the gauze and the system of plates or inside the ionization chamber. The experiments were carried out at pressures ranging between 0.01 mm. to 4 mm. in air and between 0.05 mm. to 10 mm. in hydrogen.

2. When now a discharge is passed between the system of plates and the electrode R large currents are detected in the ionization chamber in spite of the p. d. applied between the plates. These currents are usually larger when the system of plates is used as a cathode in the discharge through the bell-jar, than when it is an anode. They are not proportional to the discharge currents through the bell-jar, increasing more rapidly than the latter. Thus, in one set of experiments in which the pressure was kept constant at 0.4 mm., discharge currents of one, ten and twenty-five

milliamperes produced in the ionization chamber currents of 8.10^{-8} Å, 2.10^{-6} Å and 8.10^{-6} Å, respectively. A peculiar feature of these currents is lack of saturation. The current first increases rapidly with the p. d. in the ionization chamber, with higher p. d. the increase slows down as in case of ordinary ionization currents, but even with a p. d. as high as 100 v. the current is usually 15–20% higher than with a p. d. of 50 v. The lack of saturation is equally well marked in large currents of the order of 10^{-5} Å and in small currents of 10^{-9} Å.

Interesting results were obtained when the currents in the ionization chamber were studied as a function of the pressure, the discharge current being kept constant. At large pressures ranging between 4 mm. and 0.4 mm. in air the ionization current does not change with the pressure and is comparatively small, not exceeding 10^{-8} Å when the discharge current is of the order of 10^{-2} Å. At a pressure of 0.4 mm. the ionization current begins to grow rapidly and increases more than a hundred times when the pressure is reduced from 0.4 mm. to 0.2 mm. With further reduction of the pressure the ionization current slightly decreases.

3. The origin of the observed ionization currents cannot be ascribed to the photoelectric effect caused by the radiations from the discharge. First, experiments have shown that the currents do not depend appreciably on the nature of the electrodes in the ionization chamber, copper, aluminum and graphite giving the same results. Second, the currents reaching in some experiments the magnitude of 10^{-5} Å are by far too large to be caused by the photoelectric effect, especially if we bear in mind that the radiations were cut down to less than one per cent by passing through the long and narrow spaces between the plates. Moreover, in some experiments a grounded sheet of aluminum foil was placed between the system of plates and the electrode R and the discharge passed between the later electrode and the aluminum foil. The system of plates in this case was screened from the discharge space by the foil, but large currents of the order of 10^{-7} Å could still be detected in the ionization chamber.

The currents in the ionization chamber could be caused by canal rays or presumably also by high speed negative ions which were not absorbed by the p. d. between the plates owing to the well-known phenomenon of "Umladung," consisting in the fact that the ions in canal rays lose and gain their charge in rapid succession and thus remain uncharged a considerable part of their way.[1] For this reason the p. d. across the plates was kept between 10 and 20 times as high as was necessary to absorb ions under normal conditions. Other considerations show conclusively that under large pressures at least the observed currents could not be due to canal rays or negative ions penetrating into the ionization chamber, even if we assume that the ions constantly remain uncharged during their passage between the plates. As mentioned above, considerable currents

are observed in the ionization chamber at pressures as high as 4 mm. in air. From experiments in radioactive recoil[2] it is known that recoil atoms of RaB from RaA having a velocity of 4.10^7 cm./sec. are absorbed by a layer of 15 mm. of air at 4 mm. pressure. Now the velocity of positive ions at this pressure may be calculated from their mobility, and it can be easily seen that the velocity of the ions under the experimental conditions was much lower than 4.10^7 cm./sec. Consequently they could not penetrate the layer of 5 cm. of air separating the ionization chamber from the discharge space, even assuming that they were not affected by the electric field between the plates.

It may further be objected that sheaths of ions forming on the surface of the plates cause polarization of the electric field between them, and thus allow the carriers of electricity to penetrate into the ionization chamber. Experiments have shown, however, that the ionization current between the plates increases largely with the discharge current through the bell-jar, indicating that polarization was comparatively small, and since, as mentioned above, the p. d. across the plates was much larger than necessary to absorb the ions, this source of error may be neglected.

4. It has been shown in the previous paragraph that the observed currents could not be due to the photoelectric effect nor to electrons or high speed positive ions penetrating into the ionization chamber. Neither could they be caused by slow ions diffusing into the chamber, since the system of plates would serve as an efficient trap for slow ions. The only way to interpret the results is to assume the existence in the gaseous discharge of latent carriers of electricity. Such carriers may remain in an electrically neutral state for a period of time sufficiently long to allow them to penetrate into the ionization chamber by the process of diffusion. The existence in the discharge of "neutral doublets" has been postulated some twenty years ago by Sir J. J. Thomson in his work on positive rays.[3] The neutral doublets are eventually dissociating into negative and positive ions by collision with gaseous molecules, and they could be used very conveniently for the interpretation of the origin and properties of the ionization currents described above, but this theory must be considered by now as antiquated.

Latent carriers of electricity, however, can now easily be introduced in terms of metastable atoms. Horton and Davies[4] have shown that metastable atoms are instrumental in carrying currents through a discharge tube. If the average life of an excited atom is not small compared with the time necessary for the atom to diffuse into the ionization chamber, a large number of metastable atoms may accumulate in the chamber. To account for currents of the order of 10^{-5} Å the concentration of metastable atoms must be exceedingly large and may be estimated in the following way. Assuming that the formation of a pair of ions in the chamber

is caused only by a mutual collision of two metastable atoms, a current of the order of 10^{-5} Å corresponds to 10^{14} collisions of this kind per sec. The total number of collisions per sec. of the gaseous molecules in the ionization chamber (a volume of 100 cm: at 0.4 mm. pressure) is of the order of 10^{24}. The concentration of metastable atoms is then $\left(\frac{10^{14}}{10^{24}}\right)^{1/2} = 10^{-5}$, or one metastable atom to 10^5 gaseous molecules. The concentration of metastable atoms in the discharge space is still higher, since a number of them must have vanished before they had reached the ionization chamber. It can be shown, however, that this is in good agreement with the experimental data on metastable atoms. On the assumption that in the discharge metastable atoms are formed at about the same rate as ions, a discharge current of 10^{-2} Å is followed by the formation of 10^{17} metastable atoms per second. Allowing 10^{-3} sec. for their average life, the number of metastable atoms present in the discharge is of the order of 10^{14}. Since the total number of gaseous molecules in the discharge space under the experimental conditions was considerably less than 10^{19}, the concentration of metastable atoms was higher than 10^{-5}.

Thus, the origin of the observed currents can be ascribed to metastable atoms diffusing into the ionization chamber. The fact mentioned above that these currents increase more rapidly than the discharge currents producing them is in good agreement with the theory, since the number of collisions between metastable atoms increases as the square of the number of atoms. It would be difficult, however, to interpret some other properties of the currents, for instance, the lack of saturation or the sudden increase in the current when the pressure is reduced below a certain value.

The experiments described above may throw some light on the complicated phenomena observed behind a perforated cathode and usually ascribed to the high speed positive ions passing through the aperture in the cathode. In fact, when the system of plates is a cathode in the discharge and the pressure is considerably reduced, the apparatus may be used for the study of canal rays. It has been shown, however, that the currents observed under these conditions in the ionization chamber are only partly caused by the swift positive ions. It is of interest to note that lack of saturation, which is one of the properties of these currents, is observed also in the study of canal rays.

If metastable atoms are formed in the discharge at the same rate as ions they may carry a considerable part of the current, since at large concentrations mutual collisions between them are sufficiently frequent. The greatest difficulty in the theory of the discharge is the necessity to account for the space charge produced by the electric carriers. The presence in the discharge of electrically neutral latent carriers which are

able, in virtue of their thermal agitation, to carry large currents without producing a space charge, could greatly facilitate the interpretation of the discharge of electricity through gases.

The author is indebted to Professor Bergen Davis and Professor H. W. Webb for their kind permission to use the laboratory.

[1] Wien, W., *Kanalstrahlen.* Wien, W., and Harms, F., *Handbuch der Experimentalphysik.*, **14**, Leipzig, 1927.

[2] Wertenstein, L., *Le Radium*, **7** (1910), p. 288.

[3] Thomson, J. J., *Phil. Mag.*, **18** (1909), p. 821.

[4] Horton and Davies, *Phil. Mag.*, **42** (1921), p. 746.

GRAVITATION AND THE ELECTRON[1]

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The Problem.—The translation of Dirac's theory of the electron into general relativity is not only of formal significance, for, as we know, the Dirac equations applied to an electron in a spherically symmetric electrostatic field yield in addition to the correct energy levels those—or rather the negative of those—of an "electron" with opposite charge but the same mass. In order to do away with these superfluous terms the wave function ψ must be robbed of one of its pairs ψ_1^+, ψ_2^+; ψ_1^-, ψ_2^- of components.[2] These two pairs occur unmixed in the action principle except for the term

$$m(\psi_1^+ \bar{\psi}_1^- + \psi_2^+ \bar{\psi}_2^- + \psi_1^- \bar{\psi}_2^+ + \psi_2^- \bar{\psi}_2^+) \qquad (1)$$

which contains the mass m of the electron as a factor. But mass is a gravitational effect: it is the flux of the gravitational field through a surface enclosing the particle in the same sense that charge is the flux of the electric field.[3] In a satisfactory theory it must therefore be as impossible to introduce a non-vanishing mass without the gravitational field as it is to introduce charge without electromagnetic field. It is therefore certain that the term (1) can at most be right in the large scale, but must really be replaced by one which includes gravitation; this may at the same time remove the defects of the present theory.

The direction in which such a modification is to be sought is clear: the field equations arising from an action principle[4]—which shall give the true laws of interaction between electrons, protons and photons only after quantization—contain at present only the Schrödinger-Dirac quan-

tity ψ, which describes the wave field of the *electron*, in addition to the four potentials φ_p of the electromagnetic field. It is unconditionally necessary to introduce the wave field of the *proton* before quantizing. But since the ψ of the electron can only involve two components, ψ_1^+, ψ_2^+ should be ascribed to the electron and ψ_1^-, ψ_2^- to the proton. Obviously the present expression, $-e\cdot\widetilde{\psi}\psi$ for charge-density,[5] being necessarily negative, runs counter to this, and something must consequently be changed in this respect. Instead of one law for the conservation of charge we must have two, expressing the conservation of the number of electrons and protons separately.

If one introduces the quantities $\frac{e\varphi_p}{ch}$ instead of φ_p (and calls them φ_p), the field equations contain only the following combinations of atomistic constants: the pure number $\alpha = e^2/ch$ and h/mc, the "wave-length" of the electron. Hence the equations certainly do not alone suffice to explain the atomistic behavior of matter with the definite values of e, m and h. But the subsequent quantization introduces the quantum of action h, and this together with the wave-length h/mc will be sufficient, since the velocity of light c is determined as an absolute measure of velocity by the theory of relativity.

The introduction of the atomic constants by the quantum theory—or at least that of the wave-length—into the field equations has removed the support from under my principle of gauge-invariance, by means of which I had hoped to unify electricity and gravitation. But as I have remarked,[6] it possesses an equivalent in the field equations of quantum theory which is its perfect counterpart in formal respects: the laws are invariant under the simultaneous substitution of $e^{i\lambda}\cdot\psi$ for ψ and $\varphi_p - \frac{\partial\lambda}{\partial x_p}$ for φ_p, where λ is an arbitrary function of position in space and time. The connection of this invariance with the conservation law of electricity remains exactly as before: the fact that the action integral is unaltered by the infinitesimal variation

$$\delta\psi = i\,\lambda\cdot\psi, \qquad \delta\varphi_p = -\frac{\partial\lambda}{\partial x_p}$$

(λ an arbitrary infinitesimal function) signifies the identical fulfilment of a dependence between the material and the electromagnetic laws which arise from the action integral by variations of the ψ and φ, respectively; it means that the conservation of electricity is a double consequence of them, that it follows from the laws of matter as well as electricity. This new principle of gauge invariance, which may go by the same name, has the character of general relativity since it contains an arbitrary function λ, and can certainly only be understood with reference to it.

It was such considerations as these, and not the desire for formal generalizations, which led me to attempt the incorporation of the Dirac theory into the scheme of general relativity. We establish the metric in a world point P by a "Cartesian" system of axes (instead of the g_{pq}) consisting of four vectors $\mathbf{e}(\alpha)$ $\{\alpha = 0, 1, 2, 3\}$ of which $\mathbf{e}(1)$, $\mathbf{e}(2)$, $\mathbf{e}(3)$ are real space-like vectors while $\mathbf{e}(0)/i$ is a real time-like vector of which we expressly demand that it be directed toward the future. A rotation of these axes is an orthogonal or Lorentz transformation which leaves these conditions of reality and sign unaltered. The laws shall remain invariant when the axes in the various points P are subjected to arbitrary and independent rotations. In addition to these we need four (real) coördinates x_p $(p = 0, 1, 2, 3)$ for the purpose of analytic expression. The components of $\mathbf{e}(\alpha)$ in this coördinate system are designated by $e^p(\alpha)$. We need such local cartesian axes $\mathbf{e}(\alpha)$ in each point P in order to be able to describe the quantity ψ by means of its components ψ_1^+, ψ_2^+; ψ_1^-, ψ_2^-, for the law of transformation of the components ψ can only be given for orthogonal transformations as it corresponds to a representation of the orthogonal group which cannot be extended to the group of all linear transformations. The tensor calculus is consequently an unusable instrument for considerations involving the ψ.[7] In formal aspects our theory resembles the more recent attempts of Einstein to unify electricity and gravitation.[8] But here there is no talk of "distant parallelism"; there is no indication that Nature has availed herself of such an artificial geometry. I am convinced that if there is a physical content in Einstein's latest formal developments it must come to light in the present connection. It seems to me that it is now hopeless to seek a unification of gravitation and electricity without taking material waves into account.

Use of the Indices.—If $t(\alpha)$ be the components of an arbitrary vector at point P with respect to the axes $\mathbf{e}(\alpha)$, then

$$t^p = \sum_\alpha e^p(\alpha)\, t(\alpha) \qquad (2)$$

are its contravariant components in the coördinate system x_p. Conversely, from the covariant components t_p referred to the coördinates one obtains the components $t(\alpha)$ along the axes by the equations

$$t(\alpha) = \sum_p e^p(\alpha) t_p. \qquad (3)$$

Equations (2), (3) regulate the transition from one kind of indices to the other (Greek indices referring to the axes, Latin sub- or superscripts to coördinates.) In the inverse transitions the quantities $e_p(\alpha)$, which are defined by

$$e_p(\alpha)\, e^q(\alpha) = \delta_p^q$$

and which also satisfy

$$e_p(\alpha)\, e^p(\beta) = \delta(\alpha, \beta)$$

occur as coefficients. The Kronecker δ is 1 or 0 according to whether its indices agree or not.

Symmetry and Conservation of the Energy Density.—The invariant action

$$\int \mathbf{H}\, dx \qquad (dx = dx_0\, dx_1\, dx_2\, dx_3)$$

contains *matter* (in the extended sense) and *gravitation*, the first being represented by the ψ and possibly such additional quantities as the electromagnetic potentials φ_p, the latter by the components $e^p(\alpha)$ of the $\mathbf{e}(\alpha)$. Variation of the first kind of quantities gives rise to the equations of matter, variation of the $e^p(\alpha)$ to the gravitational equations. We disregard for the present that part of the action which depends only on the $e^p(\alpha)$, as introduced by Einstein in his classical theory of gravitation (1916), and consider only that part $\mathbf{H}$ which occurs even in the special theory of relativity. By an arbitrary infinitesimal variation of the $e^p(\alpha)$ which shall vanish outside of a finite portion of the world, an equation

$$\delta \int \mathbf{H} dx = \int \mathbf{t}_p(\alpha)\, \delta e^p(\alpha) \cdot dx \qquad (4)$$

is obtained which defines the components $\mathbf{t}_p(\alpha)$ of the "energy density." In consequence of the equations of matter, which are assumed to hold, it is immaterial if or how the quantities describing matter are varied. Because of the invariance of the action (4) must vanish for variations $\delta e^p(\alpha)$ obtained by 1) subjecting the axes $\mathbf{e}(\alpha)$ to an infinitesimal rotation which may depend arbitrarily on position and 2) subjecting the coordinates x_p to an arbitrary infinitesimal transformation, the axes $\mathbf{e}(\alpha)$ being unaltered. The first process is described by

$$\delta e^p(\alpha) = o(\alpha\beta) e^p(\beta)$$

where $o(\alpha\beta)$ constitute an anti-symmetric matrix whose elements are arbitrary (infinitesimal) functions of position. This requirement yields the symmetry law:

$$\mathbf{t}_p(\beta) e^p(\alpha) = \mathbf{t}(\alpha, \beta)$$

depends symmetrically on the two indices, α and β. But it must be observed that this law is not identically fulfilled, as in the old field theory, but only in consequence of the equations of matter, for if the wave field ψ be held unchanged the components ψ_ρ must undergo a transformation which is induced by the rotation of the axes. If $\delta x_p = \xi^p$ be the change which the coördinates of point P undergo in the second process, then the components of the unaltered vector $\mathbf{e}(\alpha)$ in P will undergo the change

$$\delta' e^p(\alpha) = \frac{\partial \xi^p}{\partial x_q} \cdot e^q(\alpha).$$

This must, on the other hand, be given by

$$\delta e^p(\alpha) + \frac{\partial e^p(\alpha)}{\partial x_q}\xi^q$$

where δ means the difference at two points P which have the same values x_p of coördinates before and after the deformation. From this there arises in the usual way[9]—again assuming the validity of the equations of matter—the differential quasi-conservation law for energy (and linear momentum) whose four components are

$$\frac{\partial \mathfrak{t}_p^q}{\partial x_q} + \mathfrak{t}_q(\alpha)\frac{\partial e^q(\alpha)}{\partial x_p} = 0. \qquad (5)$$

(Only in the special theory of relativity, where the second member is lacking, is it a true conservation law.)

It is not necessary that the integral of $\mathbf{H}$ be invariant, but only that its variation be. This is the case when $\mathbf{H}$ differs from a scalar density by a divergence; we then say that the integral is "practically invariant." Similarly it is only necessary that it be "practically real," i.e., that the difference between $\mathbf{H}$ and its complex conjugate be a divergence.

Gradient of ψ.—Let the wave field ψ be given. The invariant change $\delta\psi$ of ψ on going from the point P to a neighboring point P' is to be determined as follows. The axes $\mathbf{e}(\alpha)$ in P are taken to P' by parallel displacement: $\mathbf{e}'(\alpha)$. $\psi_\rho = \psi_\rho(P)$ being the components of ψ with respect to the axes $\mathbf{e}(\alpha)$ at P, let ψ'_ρ be the components of ψ in P' relative to this displaced system: $\delta\psi_\rho = \psi'_\rho - \psi_\rho$. These $\delta\psi_\rho$ depend only on the choice of axes in P and transform in the same way as the ψ_ρ on rotation of these axes. The axes $\mathbf{e}'(\alpha)$ are obtained from the $\mathbf{e}(\alpha)$ in P' by an infinitesimal orthogonal transformation $do(\alpha\beta)$; consequently the ψ'_ρ are obtained from the components $\psi_\rho(P')$ by the corresponding linear transformation[10] dE and we have, $d\psi_\rho$ being the differential $\psi_\rho(P') - \psi_\rho(P)$:

$$\delta\psi = d\psi + dE.\psi.$$

If $\mathbf{e}(\alpha)$ be taken as the vector $\overrightarrow{PP'}$ (multiplied by an infinitesimal factor) we write (on ignoring this factor) $o(\alpha,\beta\gamma)$, $E(\alpha)$, $\psi(\alpha)$ in place of $do(\beta\gamma)$, dE, $\delta\psi$:

$$\psi(\alpha) = \left(e^p(\alpha)\frac{\partial}{\partial x_p} + E(\alpha)\right)\psi \quad \text{or} \quad \psi_p = \left(\frac{\partial}{\partial x_p} + E_p\right)\psi.$$

The calculation of o is accomplished by means of the formula

$$e^p(\gamma)\{o(\alpha,\beta\gamma) + o(\beta,\gamma\alpha)\} = \frac{\partial e^p(\alpha)}{\partial x_q}e_q(\beta) - \frac{\partial e^p(\beta)}{\partial x_q}e_q(\alpha).$$

The right-hand side of this expression is the "commutator product" of the two vector fields $\mathbf{e}(\alpha)$ and $\mathbf{e}(\beta)$, an invariant (under transformations of the coördinates x_p), known from the Lie theory.

Introduction of Dirac's Action.—Let $S(\alpha)$ denote linear transformations which transform ψ_1^+, ψ_2^+ and ψ_1^-, ψ_2^- among themselves. They are described by the matrices[11]

$$S(0) = \begin{Vmatrix} i & 0 \\ 0 & i \end{Vmatrix}, \quad S(1) = \begin{Vmatrix} 0 & i \\ 1 & 0 \end{Vmatrix}, \quad S(2) = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}, \quad S(3) = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

for ψ_1^+, ψ_2^+; for ψ_1^-, ψ_2^- the expression for $S(0)$ is unchanged, $S(1)$, $S(2)$ and $S(3)$ assume the opposite sign. The essential fact is that the quantities

$$\tilde{\psi}' S(\alpha)\psi$$

transform like the four components $t(\alpha)$ of a vector on rotation of the axes $\mathbf{e}(\alpha)$, ψ' being a quantity of the same kind as ψ. (In particular, $\tilde{\psi} S(\alpha)\psi$ is the four-vector flux of probability.) Therefore

$$\tilde{\psi} S(\alpha)\psi(\alpha) = \tilde{\psi} e^p(\alpha) S(\alpha) \frac{\partial \psi}{\partial x_p} + \tilde{\psi} S(\alpha) E(\alpha) \psi$$

is a scalar, and after dividing by the absolute value ϵ of the determinant

$$| e^p(\alpha) |$$

we obtain a scalar density $i\mathfrak{H}$ whose integral can be employed as action. Division by ϵ will be indicated by changing the ordinary letter into the corresponding gothic. The calculation yields

$$\frac{1}{\epsilon} S(\alpha) E(\alpha) = \frac{1}{2} \frac{\partial \mathfrak{e}^p(\alpha)}{\partial x_p} S(\alpha) + \frac{1}{2} \mathfrak{I}(\alpha) S'(\alpha),$$

where $S'(\alpha)$ is a transformation analogous to $S(\alpha)$: it agrees with $S(\alpha)$ for ψ_1^+, ψ_2^+, but is $-S(\alpha)$ for ψ_1^-, ψ_2^-.

$$\begin{aligned} \epsilon \mathfrak{I}(\alpha) &= o(\beta,\gamma\delta) + o(\gamma,\delta\beta) + o(\delta,\beta\gamma) \\ &= \Sigma \pm \frac{\partial e^p(\beta)}{\partial x_q} e_q(\gamma)\, e^p(\delta) \end{aligned}$$

where the summation (in addition to that over p and q) is alternating and extends over the six permutations of $\beta\gamma\delta$ while $\alpha\beta\gamma\delta$ is an even permutation of the indices 0, 1, 2, 3.

We have yet to investigate whether $\mathfrak{H}$ is practically real. Since

$$S^p = e^p(\alpha) S(\alpha) \quad \text{and} \quad S(\alpha) E(\alpha)$$

are Hermitean matrices

$$-i\overline{\mathfrak{H}} = \frac{1}{\epsilon} \left\{ \frac{\partial \tilde{\psi}}{\partial x_p} e^p(\alpha) S(\alpha)\psi + \tilde{\psi} S(\alpha) E(\alpha) \psi \right\}.$$

The first part is, on neglecting a complete divergence,

$$-\tilde{\psi}\frac{\partial(\mathbf{e}^{\flat}(\alpha)S(\alpha)\psi)}{\partial x_{\flat}} = -\tilde{\psi}\,\mathbf{e}^{\flat}(\alpha)S(\alpha)\frac{\partial\psi}{\partial x_{\flat}} - \frac{\partial\mathbf{e}^{\flat}(\alpha)}{\partial x_{\flat}}.\tilde{\psi}S(\alpha)\psi.$$

On adding and subtracting $\mathbf{H}$ and $\overline{\mathbf{H}}$ we obtain the two action quantities $\mathbf{m}$, $\mathbf{m}'$ ($\mathbf{m}$ = matter):

$$i\mathbf{m} = \tilde{\psi}\mathfrak{S}^{\flat}\frac{\partial\psi}{\partial x_{\flat}} + \frac{1}{2}\frac{\partial\mathbf{e}^{\flat}(\alpha)}{\partial x_{\flat}}.\tilde{\psi}S(\alpha)\psi \qquad (6)$$

and

$$\mathbf{m}' = \mathbf{I}(\alpha).\tilde{\psi}S'(\alpha)\psi. \qquad (7)$$

Both are practically, not actually, invariant, the first is practically and the second actually real.

The first is the essential content of the Dirac theory, written in general invariantive form. The corresponding tensor density of energy

$$\mathbf{t}_{\flat}(\alpha) = \mathfrak{s}_{\flat}(\alpha) - e_{\flat}(\alpha)\mathfrak{s}$$

where

$$\epsilon\mathfrak{s}_{\flat}(\alpha) = \frac{1}{2i}\left\{\tilde{\psi}S(\alpha)\frac{\partial\psi}{\partial x_{\flat}} - \frac{\partial\tilde{\psi}}{\partial x_{\flat}}S(\alpha)\psi\right\}$$

and $\mathfrak{s}$ the contracted

$$\mathfrak{s}_{\flat}(\alpha)\ e^{\flat}(\alpha).$$

It has already been given in the literature for the Dirac theory (special relativity). For the electron of hydrogen in the normal state we find that the integral

$$\int \mathbf{t}_0^0\,dx_1\,dx_2\,dx_3$$

extended over a section $x_0 = t$ = const., which should yield the mass, has the value $m/\sqrt{1-\alpha^2}$ (α the fine structure constant); this is a reasonable result, since m is to be taken as proper mass and in the Bohr theory αc is the velocity of the electron in the normal state.

It is worthy of note that there occurs in addition the action $\mathbf{m}'$, which is unknown to special relativity since it vanishes for constant $e^{\flat}(\alpha)$.

The Electromagnetic Field.—In the Dirac theory the influence of an electromagnetic field is taken into account by replacing the operator $\frac{\partial}{\partial x_{\flat}}$, affecting the ψ by $\frac{\partial}{\partial x_{\flat}} + i\varphi_{\flat}$. This yields an additional term of the form $i\varphi(\alpha)\cdot\tilde{\psi}S(\alpha)\psi$ in the action. On comparing this with (6) and (7) one might think that $\epsilon.\varphi(\alpha)$ is to be identified with $\partial\mathbf{e}^{\flat}(\alpha)/\partial x_{\flat}$ or $\mathbf{I}(\alpha)$. Disregarding the material waves ψ one would then have a theory of electricity of the same kind as the latest development of Einstein; the φ are expressed in terms of the $e^{\flat}(\alpha)$ in a way which is invariant under transformations of the coördinates, but only permit the cogredient rotations of the axes in all points P (distant parallelism). However, I believe

one must proceed otherwise in order to bring in the electromagnetic field. We have previously not mentioned the fact that the linear transformation of the ψ corresponding to a rotation of axes is not uniquely determined. It is indeed possible to normalize this transformation, and we have tacitly based our calculations on such a normalization; but the normalizing is itself a double-valued process and this reveals its artificial character. The mathematical connection is this: we are to find a linear transformation of the four components of ψ such that the quantities

$$\tilde{\psi} S(\alpha) \psi \tag{8}$$

suffer the given rotation.[12] Obviously it remains unaltered when ψ_1^+, ψ_2^+ are multiplied by $e^{i\lambda^+}$, ψ_1^-, ψ_2^- by $e^{i\lambda^-}$ (transformation L) where λ^+ and λ^- are arbitrary real numbers. It is readily seen that the transformations L are the only ones which induce the identical transformation of the quantities (8); i.e., which satisfy the four equations

$$\tilde{L} S(\alpha) L = S(\alpha).$$

A transformation of the four components of ψ which multiplies ψ_1^+, ψ_2^+ by a number a^+ and ψ_1^-, ψ_2^- by a number a^- will be called *spinless quantity* $A = [a^+, a^-]$.

In consequence of this the dE employed above is only determined to within the addition of an arbitrary spinless imaginary quantity idF. Hence we obtain an additive term

$$iF(\alpha) . \tilde{\psi} \mathbf{S}(\alpha) \psi$$

in the action $\mathbf{m}$, in which the $F(\alpha)$ are real spinless quantities $[\varphi^+(\alpha), \varphi^-(\alpha)]$ constituting the components of a vector with respect to the axes. $\partial/\partial x_p$ is to be replaced by $\partial/\partial x_p + iF_p$. We now employ the letter $\mathbf{m}$ to denote this completed action. The introduction of F obviously brings with it invariance under the simultaneous replacement of

$$\psi \text{ by } e^{iL}\psi \text{ and } F_p \text{ by } F_p - \frac{\partial L}{\partial x_p}$$

where L is a real spinless quantity which depends arbitrarily on position. This "gauge invariance" shows why it is impossible to employ the scalar (1) as an action function.

We must naturally interpret F_p as the components of electromagnetic potential. The two fields φ_p^+, φ_p^- are independent of the choice of the axes $\mathbf{e}(\alpha)$ except that they are interchanged on transition from right- to a left-handed system of axes.

$$\frac{\partial \varphi_q^+}{\partial x_p} - \frac{\partial \varphi_p^+}{\partial x_q} = \varphi_{pq}^+ \text{ and the corresponding } \varphi_{pq}^-$$

are gauge-invariant anti-symmetric tensors.

In analogy to the Maxwellian action quantity, and in accordance with the above properties, the two functions $\mathbf{f}$, $\mathbf{f}'$ ($\mathbf{f}$ = electromagnetic field) defined by

$$\epsilon\mathbf{f} = \varphi^{+}_{pq}\,\varphi^{pq}_{-} \text{ and } \epsilon\mathbf{f}' = \varphi^{+}_{pq}\,\varphi^{pq}_{+} + \varphi^{-}_{pq}\,\varphi^{pq}_{-} \tag{9}$$

are to be considered in choosing the scalar density of electromagnetic action. The identification of F with the electromagnetic potential is then justified by the fact that the entity which is represented by F influences and is influenced by matter in exactly the same way as the electromagnetic potentials. *If our view is correct, then the electromagnetic field is a necessary accompaniment of the matter-wave field and not of gravitation.*

If the action, insofar as its dependence on the ψ and φ is concerned, is an additive combination of $\mathbf{m}$, $\mathbf{m}'$, $\mathbf{f}$, $\mathbf{f}'$ we obtain the two conservation theorems

$$\frac{\partial \rho^{p}_{+}}{\partial x_p} = 0 \text{ and } \frac{\partial \rho^{p}_{-}}{\partial x_p} = 0 \tag{10}$$

where

$$\rho^{p}_{+} = \tilde{\psi}^{+} S^{p} \psi^{+}$$

contains only ψ^{+}_{1} and ψ^{+}_{2} and similarly for ρ^{p}_{-}. They are a double consequence of the field laws, that is, of the equations of matter in the narrow sense as well as of the electromagnetic equations. These two identities obtaining thus between the field equations are an immediate consequence of the gauge invariance. In consequence of (10) the two integrals n^{+} and n^{-} of $\rho = \rho^{0}$:

$$n = \int \rho\, dx_1 dx_2 dx_3$$

which are to be extended over a section $x_0 = \text{const.}$ are invariants which are independent of the "time" x_0.[13] Since they are interchanged on transition from right- to left-handed axes their values, which are absolute constants of nature, must be equal if both kinds of axes are to be equally permissible. We normalize them, in accordance with the interpretation of ψ as probability, by

$$n^{+} = 1, \quad n^{-} = 1. \tag{11}$$

We have already mentioned how the quasi-conservation laws for energy, momentum and moment of momentum are related to invariance under transformation of coördinates and rotation of axes $\mathbf{e}(\alpha)$.

A stationary solution is characterized by the fact that $e^{p}(\alpha)$ and $F(\alpha)$ are independent of time $x_0 = t$, while the ψ^{+} contain the time in an exponential factor $e^{i\nu t}$, the ψ^{-} in a factor $e^{i\nu' t}$; ν and ν' need not be equal.

Gravitation.—We consider as the gravitational part of the action the practically (not actually) invariant density $\mathbf{g}$ ($\mathbf{g}$ = gravitation) which underlies Einstein's "classical" theory of gravitation and which depends only on the $e^{p}(\alpha)$ and their first derivatives. It is most appropriate to

carry through anew the entire calculation, which leads through the Riemann curvature tensor, in terms of the $e^{\flat}(\alpha)$; we find

$$\epsilon\mathfrak{g} = o(\alpha, \alpha\gamma)\, o(\beta, \beta\gamma) + o(\alpha, \beta\gamma)\, o(\beta, \alpha\gamma).$$

The "cosmological term" $\mathfrak{g}'$ is given by $\epsilon\mathfrak{g}' = 1$.

If gravitation be represented by $\mathfrak{g}$, one can, as is well known, add to the material + electromagnetic energy $\mathfrak{t}_{\flat}^{q}$ a gravitational energy $\mathfrak{v}_{\flat}^{q}$ in such a way that the sum satisfies a true conservation law.[14] Designating the total differential of $\mathfrak{g}$ considered as a function of $e^{\flat}(\alpha)$ and $e_{q}^{\flat}(\alpha) = \frac{\partial e^{\flat}(\alpha)}{\partial x_{q}}$ by

$$\delta\mathfrak{g} = \mathfrak{g}_{\flat}(\alpha)\delta e^{\flat}(\alpha) + \mathfrak{g}_{\flat}^{q}(\alpha)\delta e_{q}^{\flat}(\alpha),$$

then

$$-\mathfrak{v}_{\flat}^{q} = \mathfrak{g}_{r}^{q}(\alpha)\frac{\partial e^{r(\alpha)}}{\partial x_{\flat}} - \delta_{\flat}^{q}\mathfrak{g}.$$

We obtain thus an invariant constant mass m which must be one of the characteristic universal constants of Nature.

Doubts, Prospects.—(11) is to be interpreted as the law of conservation of the number (or charge) of electrons and protons. Therefore we ascribe ψ^{+} and ψ^{-} to the electron and to the proton, respectively. Taking $\mathfrak{f}$ as the electromagnetic part of the action, which seems plausible to me, we then obtain Maxwell's equations in the sense that the proton generates the field φ^{+} and the electron φ^{-}; whereas in accordance with the equations of matter φ^{+} will effect only the electron and φ^{-} the proton. This is not as obstruse as it may sound; on the contrary, the previous theory leads to entirely false results if the potential due to the electron, which at large distances neutralizes that due to the nucleus, reacts on the electron itself, as Schrödinger has pointed out with emphasis.[15] It may indeed seem queer that ψ^{+} and ψ^{-} are here equally permissible, since we know that positive and negative electricity are fundamentally different—that protons and electrons have different mass. But if we neglect the gravitational and electrical energy in comparison with the material the mass m falls into two parts m^{+} and m^{-} which are, however, not strictly constant. It is possible that m^{+} and m^{-} are different if our equations admit two classes of solutions which are interchanged on transition from right- to left-handed axes—as in the Dirac theory, the spherically symmetric hydrogen problem admits several solutions for the normal state which are not themselves spherically symmetric but which are transformed among themselves by rotation.

As far as we know $\mathfrak{m}$, $\mathfrak{g}$ and one of the two quantities $\mathfrak{f}$ or $\mathfrak{f}'$ are indispensable for the explanation of the phenomena. I am inclined to believe that the action is composed additively of $\mathfrak{m}$, $\mathfrak{f}$ and $\mathfrak{g}$.

It should be noted that our field equations contain neither the theory

of a single electron nor that of a single proton. One might rather consider them as the laws governing a hydrogen atom consisting of an electron and a proton; but here again, the problem of interaction between the two may first require quantization. What we have obtained is solely a field scheme which can only be applied to and compared with experience after the quantization has been accomplished. We know from the Pauli exclusion principle[16] what commutation rules are to be applied in the quantization of ψ^+; those for ψ^- must be the same in our theory. The commutation relations between ψ^+ and ψ^- are as yet entirely unknown. Those of the electromagnetic field (photons) are almost completely known. In this respect we know nothing concerning the gravitational field. The commutation rules for F are here almost completely fixed by those for ψ by the condition that these latter be unaltered when ψ is given the increment $\delta\psi = iF(\alpha)\psi$. That the rules thus obtained are in agreement with experience is indeed a support for our theory; i.e., it tells us why the "anti-symmetric," Pauli-Fermi statistics for electrons leads to the "symmetric" Bose-Einstein statistics for photons. A definite decision can, however, first be reached when the barrier which hems the progress of quantum theory is overcome: the quantization of the field equations.

[1] I am indebted to Prof. H. P. Robertson for the translation of my German manuscript.

[2] I employ the same notation as in my book *Gruppentheorie und Quantenmechanik*, Leipzig, **1928** (cited as *G Q*) except that I here write $\psi_1^+ \psi_2^+, \psi_1^- \psi_2^-$ in place of $\psi_1 \psi_2, \psi_3 \psi_4$. Cf. in particular § 25, 39, 44. $\frac{h}{2\pi}$ is Planck's constant.

[3] Cf. H. Weyl, *Space—Time—Matter*, London, **1922** (cited as *S T M*), § 33.

[4] *G Q*, pp. 199, 200.

[5] The circumflex indicates transition to the conjugate of the transposed matrix (Hermitean conjugate). The four components of ψ are considered as the elements of a matrix with four rows and one column.

[6] *G Q*, p. 88.

[7] Attempts to employ only the tensor calculus have been made by Tetrode (*Z. Physik*, **50**, 336 (1928)); J. M. Whittaker (*Proc. Camb. Phil. Soc.*, **25**, 501 (1928)), and others; I consider them misleading.

[8] *Sitzungsber. Berl. Akad.*, **1928**, pp. 217, 224; **1929**.

[9] Cf. the analogous considerations in *S T M*, pp. 233–237.

[10] Capital Latin letters (except P for point) denote linear transformations of the four components of ψ.

[11] In *G Q*, loc. cit., they are denoted by s'_α.

[12] It is to be borne in mind that under the influence of a proper Lorentz transformation the ψ^+ components — as well as the ψ^- — are transformed among themselves. Only when the improper operations of the Lorentz group, the reflection

$$\mathbf{e}(0) \to \mathbf{e}(0), \qquad \mathbf{e}(\alpha) \to -\mathbf{e}(\alpha) \quad [\alpha = 1, 2, 3],$$

is taken into account is it necessary to use both pairs of components together.

[13] *S T M*, p. 289.

[14] The derivation in *S T M*, pp. 269, 270, can be adapted to the new analytic formulation of the gravitational field.

[15] E. Schrödinger, *Ann. Physik*, **82**, 265–272 (1927); in particular p. 270.

[16] P. Jordan and E. Wigner, *Z. Physik*, **47**, 1928, 631; G Q, § 44.

Correction made on the proofs (March 4, (1929)).—The calculation of the action density **m** contains an error which should be corrected as follows. The spacial components $\alpha = 1,2,3$ of $l(\alpha)$ are pure imaginary and the temporal component $l(0)$ real, not the opposite as I had assumed. In the definition of $\mathbf{m}'$ we must therefore divide the right-hand side by $2i$. But this has as consequence that the $\mathbf{H} = \mathbf{m} + \mathbf{m}'$ obtained from the calculation is practically real and not composed of a real and an imaginary part. We therefore obtain but *one* invariant action density for matter: $\mathbf{m} + \mathbf{m}'$. To the tensor density of energy arising from **m** must naturally be added the term arising from $\mathbf{m}'$.

RAMAN SPECTRA OF SOLUTIONS OF SOME IONIZED SUBSTANCES

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A new type of secondary radiation has recently been found[1] to be present in the light scattered by liquids. If a medium is illuminated with a monochromatic radiation for which it is largely transparent, the scattered light is found to contain in addition to light of the original frequency, lines of altered frequency which do not arise from fluorescence. A characteristic of this phenomenon, the Raman Effect, is that the changes in frequency on scattering are independent of the frequency of the incident radiation. The changes in frequency are, however, specifically dependent on the nature of the scattering substance and, indeed, are often found to agree with previously known infra-red frequencies of the substance. There is little doubt that the frequency changes on scattering measure differences in the energy contents of stationary states of the molecules of the scattering substance.

The change in frequency accompanying the scattering of light has been measured for the case of the crystal calcite, $CaCO_3$, by Wood,[2] and by Landsberg and Mandelstam.[3] Landsberg and Mandelstam found strong scattered lines corresponding to the frequency change $\Delta\nu = 1095$ cm.$^{-1}$, and somewhat weaker ones corresponding to $\Delta\nu = 291$ cm.$^{-1}$; Wood reported the values 1087.0 and 280.2 for these shifts and in addition a faint shift with[4] $\Delta\nu = 730.2$ cm.$^{-1}$. In the light scattered by an aqueous solution of sodium nitrate, $NaNO_3$, Carrelli, Pringsheim and Rosen,[5] found shifted lines having $\Delta\nu = 1044$ cm.$^{-1}$.

Water has been found by various observers to give no sharp lines in the spectrum of its scattered light. The line given by sodium nitrate solution,

and also present in the spectrum[5] from nitric acid solution, is presumably to be attributed to the nitrate ion. The close agreement between the strongest lines of the Raman spectra of calcite and of sodium nitrate solution might be entirely fortuitous; however, the supposed similarity in the electronic structures of carbonate ion, $CO_3^=$, and nitrate ion, NO_3^-, as well as the similarity of these ions in masses and size, suggest that this is not the case.

In order to obtain more satisfactorily comparable data we have photographed the spectrum of the radiation scattered by an aqueous solution of potassium carbonate, K_2CO_3. The liquid was contained in a water-jacketed glass tube, and was strongly illuminated by a mercury arc, both tube and arc being placed within a cylinder nickeled on the inside. Two different spectrographs were employed—a Hilger type E2 quartz instrument and a glass instrument of lower dispersion but greater speed. The frequencies of the shifted lines were evaluated with the aid of copper arc

TABLE 1

RAMAN SPECTRA FROM $NaClO_3$ SOLUTION

FREQUENCY OF SHIFTED LINE CM.$^{-1}$	INTENSITY OF SHIFTED LINE	WAVE-LENGTH OF EXCITING LINE Å	FREQUENCY OF EXCITING LINE, CM.$^{-1}$	FREQUENCY CHANGE, CM.$^{-1}$
24226	Medium	4046.6	24705	479
22460	Medium	4358.3	22938	478
24092	Weak	4046.6	24705	613
22330	Weak	4358.3	22938	608
23772	Strong	4046.6	24705	933
22582	Weak	4077.8	24516	934
22008	Strong	4358.3	22938	930

lines recorded on the same plate. With potassium carbonate solution shifted lines corresponding to $\Delta\nu = 1063$ cm.$^{-1}$ were readily obtained; weaker lines, if present, would probably have been obliterated by the continuous background which we have thus far been unable to eliminate in the case of this solution. On the other hand, with sodium nitrate solution not only a strong shift with $\Delta\nu = 1050$ but also a weaker shift with $\Delta\nu = 717$ cm.$^{-1}$ was found.

We have also photographed Raman spectra from solutions containing the ions ClO_3^-, BrO_3^-, IO_3^-, $SO_3^=$, and HCO_3^-, as well as from solutions containing $SO_4^=$, and ClO_4^-. As regards the fainter lines of the spectra, our measurements are probably incomplete. However, since considerable time will probably be consumed before the continuous background can in all cases be suppressed, it has seemed desirable to call attention to the unexpected regularity of at least the strong lines of the spectra of all of these solutions.

Details of the assignment of exciting lines to the shifted lines are given in table 1 for the sample case of $NaClO_3$ solution. The magnitudes of

the frequency shifts together with an estimate of their relative prominence are summarized for the various substances in table 2.

From the results given in table 2 it appears that the magnitude of the frequency changes are uninfluenced by the nature of the positive ion; thus sodium, calcium, and ammonium nitrates yielded frequency changes which are identical[6] within the error of our measurements. Nitric acid solution has been found[5] to give a slightly more complicated spectrum than has been obtained with the nitrates; but, according to our observations, by far the strongest line given by nitric acid solution has $\Delta\nu = 1047$ in agreement with the nitrates. There is, moreover, considerable evidence[7] that strong nitric acid solutions contain other molecular species beside the simple ions. Although the spectra of the solutions in table 2 appear to

TABLE 2

RAMAN SPECTRA FROM VARIOUS IONIZED SUBSTANCES

SCATTERING MATERIAL	CONCENTRATION OF SOLUTION	FREQUENCY CHANGE (CM^{-1}) AND INTENSITY			OBSERVER
$CaCO_3$	Calcite crystal	1087 (v. st.)	730 (f.)	280 (st.)	Wood[2]
		1095 (v. st.)		291 (st.)	L. and M.[3]
K_2CO_3	52%	1063			D. and D.
$KHCO_3$	26%	1035			D. and D.
$NaNO_3$	47%	1050 (st.)	717 (w.)		D. and D.
$Ca(NO_3)_2$	41%	1052 (st.)	723 (w.)		D. and D.
NH_4NO_3	52%	1050 (st.)	(723) (w.)		D. and D.
K_2SO_3	52%	988			D. and D.
$(NH_4)_2SO_4$	43%	987 (st.)	624 (m.)	458 (m.)	D. and D.
$NaClO_3$	51%	932 (st.)	611 (w.)	479 (m.)	D. and D.
$HClO_4$	60%	935 (st.)	634 (m.)	470 (m.)	D. and D.
$NaBrO_3$	26%	810 (st.)		367 (m.)	D. and D.
HIO_3	45%	800 (st., broad)		329 (m., broad)	D. and D.

arise from the negative ions, it is not unlikely that other, especially more complex, positive ions, might well give rise to spectra of their own.

It may be observed that in all the cases in table 2 the spectra are characterized by one line considerably stronger than any others. For this line the frequency change is very little different for the solutions containing the ions $CO_3^=$, HCO_3^-, and NO_3^-. Also the frequency change is substantially the same for $SO_3^=$ as for $SO_4^=$, and the same for ClO_3^- as for ClO_4^-, and accordingly is in these cases, independent of the number of oxygen atoms associated with the central atom. Indeed, except for the relative intensities of the weaker lines, the spectra of sodium chlorate and perchloric acid are scarcely distinguishable. Finally (omitting the somewhat irregular case of HCO_3^-) the frequency change corresponding to the strong line decreases with increase in atomic number of the central atom of the negative ion. This fact seems rather surprising if the magnitude

of the shift can, when suitable corrected for the masses involved, be taken as a measure of the stiffness of binding.

We are indebted to Professor Ira S. Bowen for valuable advice relative to the present measurements.

[1] Raman, *Indian J. of Phys.*, **2**, 387 (1928); Raman and Krishnan, *Ibid.*, **2**, 399 (1928).

[2] Wood, *Phil. Mag.*, **6**, 742 (1928).

[3] Landsberg and Mandelstam, *Z. Phys.*, **50**, 769 (1928).

[4] There appears to be a misprint in the original paper.

[5] Carrelli, Pringsheim and Rosen, *Z. Phys.*, **51**, 511 (1928).

[6] Carrelli, Pringsheim and Rosen, loc. cit., reported somewhat lower values for ammonium nitrate (mean 1014) than for sodium nitrate.

[7] Mellor, *Treatise on Inorganic and Theoretical Chemistry*, vol. VIII, p. 566.

THE CRYSTAL STRUCTURE OF STRONTIUM

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The crystal structure of strontium has been recently determined by Simon and Vohsen[1] from the powder spectrogram obtained in an ingenious low-temperature x-ray diffraction camera which was designed for the study of the alkali metals. They have assigned a face-centered cubic lattice with a unit cube edge, $a_0 = 6.03$ A.U. The small resolution of their camera, however, limited the accuracy of their results, so that it was thought advisable to publish the results obtained under more ideal conditions in our study of the alkaline earth metals and their alloys.

Strontium was obtained for this measurement by the sublimation of the metal from a mixture of pure strontium oxide and aluminum[2] in a high vacuum at 1000°C. This product was further purified by resublimation at 700°C. in vacuum. The metal resulting from this final treatment was silvery white, quite soft and contained over 99.9 per cent strontium.

The samples for x-ray analysis were prepared by the same method as was used in the study of barium.[3]

The powder spectrograms were obtained with a quadrant camera, $r = 20.32$ centimeters, using the MoK radiation from a Coolidge tube. Due to an exceptionally strong fogging of the film, it was found necessary to introduce a heavy ZrO_2 filter into the cassette in addition to the filter which was supplied with it, since the latter was not sufficient to remove the intense secondary radiation. That this fogging is due to a secondary radiation from strontium and not to impurities is apparent from the

fact that barium of similar purity gave sharply defined diffraction lines without fogging in spite of its greater action in air. Since the wavelength of the characteristic radiation of molybdenum is so close to that of strontium, it is to be expected that a strong secondary radiation would result when the molybdenum radiation strikes strontium atoms. This is further substantiated by the fact that metallic yttrium behaves similarly under the same conditions. The atomic numbers of molybdenum, yttrium, and strontium are 42, 39 and 38, respectively.

POWDER DIFFRACTION DATA FOR STRONTIUM FOIL

MoK_∞ radiation. Quadrant cassette $r = 20.32$ cm.

LINE	PLANE HKL (N)	d_{HKL} MEASURED A.U.	d_{HKL} CALCD. A.U.	a_0 A.U.
1	111 (1)	3.51	3.5075	6.079
2	100 (2)	3.035	3.0375	6.070
3	110 (2)	2.149	2.1482	6.071
4	113 (1)	1.830	1.8316	6.069
5	111 (2)	1.755	1.7538	6.079
6	133 (1)	1.393	1.3937	6.072
7	120 (2)	1.361	1.3594	6.087
8	112 (2)	1.240	1.2400	6.075
9	111 (3)	1.169	1.1691	6.074

a_0 (average) $= 6.075 \pm 0.004$ A.U.

The diffraction data confirm the results of Simon and Vohsen that the structure of strontium is face-centered cubic or some form closely resembling this type of lattice. Instead of a unit cube edge, $a_0 = 6.03$ A.U., however, it is found that this is more nearly equal to 6.075 A.U. The discrepancy between these two values is undoubtedly due to the small resolution of their camera which did not permit an accurate measurement of the interplanar distances.

The density of strontium calculated on the basis of a unit cell containing 4 atoms and measuring 6.075 A.U. on a side is 2.58 which agrees very well with the density, 2.6, obtained by direct measurement. The distance between strontium atoms is calculated to be 4.295 A.U. which is not in harmony with the hypothesis of constant atomic radii, since the distance between strontium and fluorine atoms in SrF_2 is 2.538 A.U.

I wish to take this opportunity to express my appreciation of the kindly interest and thoughtful suggestions of Professor George L. Clark in the progress of this work.

* NATIONAL RESEARCH FELLOW.

[1] Simon, F. and Vohsen, E., *Zeit. Physik. Chem.*, **133**, 165 (1928).

[2] Danner, *J. Am. Chem. Soc.*, **46**, 2382 (1924). Ruff and Hartman, *Zeit. anorg. Chem.*, **133**, 29 (1924). Biltz and Huttig, *Ibid.*, **114**, 241 (1920). Dafert and Miklanz, *Monatsch.*, **34**, 1685 (1913). Guntz and co-workers, cf. *Bull. soc. Chim.*, **35**, 709 (1924).

[3] Clark, King and Hyde, *Proc. Nat. Acad. Soc.*, **14**, 617 (1928).

SELECTIVE SEGREGATION OF CHROMOSOMES IN MALES OF A THIRD SPECIES OF SCIARA[1]

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In previous papers[2] evidence has been presented which indicates that in *Sciara coprophila* Lint., and in *S. similans* Joh., some, if not all, of the autosomes regularly segregate in a selective, instead of random, fashion during spermatogenesis. This process is such that the maternal member of the chromosome pair regularly passes to one pre-determined pole in the spermatocyte, while the paternal member goes in the opposite direction. Owing to the inequality of the cell division and the regular elimination (degeneration) of the smaller daughter component (bud) into which the paternal chromosome passes, this chromosome is regularly eliminated and hence not transmitted by the sperms. Conversely, its maternal homologue is regularly retained and is transmitted by all, instead of only half, the sperms. The cytological peculiarities of this division (primary spermatocyte) have been discussed in the papers cited above. Since the maternal and paternal chromosomes cannot be distinguished from each other morphologically, the conclusions just stated are, of course, based largely on genetic evidence.

In addition to the type of segregation mentioned above, two other types have been detected in the species studied most (*S. coprophila*). One is exhibited by the peculiar "sex limited" chromosomes,[3] which, although paternal in origin, regularly pass to the same pole as the known maternal autosome. Thus their segregation is selective, but reversed as compared with the known paternal autosome. The third type of behavior is shown by the recently identified sex chromosomes (XY)[4] which segregate at random.

Assuming the presence and similar behavior of sex chromosomes in *S. similans*, it follows that the same three types of segregation occur also in that species.

The male chromosome group in both of the above species consists of the "sex-limited" chromosomes (found only in males) the sex chromosome pair, and three pairs of "autosomes." Only one autosomal mutant character has been studied, and hence only one pair of autosomes followed, in each species. Considering the fact that each of the three kinds (or pairs) of chromosomes, thus far followed, differs from the other two in its mode of segregation in the male, it becomes of especial interest to ascertain how the remaining two pairs (autosomes) behave. This can

only be determined finally by following all three pairs of autosomes in one species, but since mutants are especially difficult to obtain in this material, partly on account of the peculiar type of inheritance here (discussed previously) the studies on *S. coprophila* are being supplemented by observations on any mutant characters found in other species.

The character used in the present study comes in the latter category and has been studied only for the reason just mentioned. It shows the same type of selective segregation as that exhibited by the two autosomal characters discussed above, and thus adds proportionately to the probability that in these species the autosomes all behave alike.

TABLE 1

SEE TEXT FOR EXPLANATION

MATINGS GIVING NO. FUSED			MATINGS GIVING WILD-TYPE AND FUSED				
				♀		♂	
CULTURE NO.	♀	♂	CULTURE NO.	+	FU	+	FU
i 125—1	60	0					
			i 124—1	59	48	1	0
i 125—2	90	0					
			i 124—2	0	0	47	64
i 125—5	45	0					
			i 125—3	0	0	33	30
i 128—1	0	70					
			i 125—4	0	0	82	95
—2	69	0					
			i 128—4	54	57	0	0
—3	0	61					
			i 134—1	47	50	0	0
—5	0	14					
			i 134—6	20	22	0	0
—6	0	85					
			i 213	0	0	13	16
i 134—2	28	0					
—3	73	0					
—4	110	0					
—5	126	0					

Some additional interest, perhaps, attaches to the present case because the mutation evidently occurred in nature. The history and genetic behavior of the character are outlined below.

It should be noted that the selective types of segregation noted above apply only to the male. In the female, so far as known, segregation is usually a random process.

Inheritance of "Fused" in Sciara impatiens Johannsen

Description.—The recessive mutant character "fused" is a wing-vein modification. Fused wings differ from the wild type in having the two branches of the media (*M* 1–2 and *M* 3) fused throughout their length. Since, in the wild-type, the branches are prominent, in the form of a *U*

extending nearly half the length of the wing, the character is conspicuous and easily recognized. Fused varies within relatively small limits and, so far as known, never approximates the wild-type in appearance. In the heterozygous condition, however, the fused gene sometimes produces a visible effect—apparently in combination with modifying genes which alone are not potent. This effect is a delta-like thickening or partial fusion at the fork of the media. Where manifest it serves to identify the flies heterozygous for fused, but in most of the matings considered here the necessary modifiers were apparently absent, so it is not classified separately in the tables.

TABLE 2

HETEROZYGOUS FEMALES BY FUSED MALES; PAIR MATINGS

	♀		♂	
CULTURE NO.	+	FU	+	FU
i 233—1	4	8	0	1
—2	4	6	0	0
—3	29	29	0	0
—5	44	27	0	0
—6	0	0	28	26
—7	26	23	0	0
i 237—1	0	0	8	11
i 208	0	0	7	8
i 209—1	16	9	0	0
—2	44	34	0	0
—4	2	12	0	0
—5	0	0	12	17
Total	169	148	55	63

Origin and History.—From a single wild female collected by the writer in San Diego, California, 37 wild-type and 31 "delta-like" offspring were obtained—all females, since this species gives "unisexual" progenies.[5] Sixteen other females taken at the same time and place (a greenhouse) gave only wild-type offspring.

From the progeny including the "delta-like" flies thirty-three sisters were tested, sixteen of them being "delta-like" and seventeen wild-type. The "delta-like" females were mated to wild-type males from three of the sixteen other wild females mentioned above. The wild-type females were also mated to males from various sources. The offspring in both series were all wild-type, except in one lot involving crosses to males from Cold Spring Harbor, N. Y. In the latter a different type of modification was introduced which need not be considered here, since none of the subsequent matings involved these flies. Approximately sixteen hundred flies were secured in this generation (20 female progenies: 13 males progenies).

The offspring of wild-type mothers were discarded and matings made only from the offspring of the "delta-like" females. Twenty pair matings

of the latter type were made, with the results summarized in table 1. Twelve gave only wild-type (save for a few "delta-like" individuals); the remaining eight gave wild-type and fused in a ratio of approximately 1:1 (total 356 wild-type: 372 fused).

Interpreting these results, it appears that the original wild female was heterozygous for fused and hence that half of her daughters (she gave no sons) were likewise heterozygous. The latter, in turn, mated to pure wild-type males, gave offspring half of which were similarly heterozygous. Random pair matings among these offspring included some matings in which both parents were heterozygous (eight out of twenty; expectation would call for five out of twenty). With random segregation in both sexes, the progenies from such matings should include wild-type and fused in a ratio of 3:1. With selective segregation in the male, such that only the maternal gene is transmitted, the ratio should be 1:1, as actually obtained.

Supplementing the above observations other tests were made as follows:

Heterozygous Females X Fused Males.—Twelve pair matings of this type gave the results shown in Table 2, with a total of 224 wild-type: 211 fused, indicating that segregation occurs at random in the female.

Fused X Fused.—Many matings of this type were made to test the constancy of fused. The results are not tabulated, since the character proved constant and has been kept in pure stock for about fifteen generations.

Tests of Heterozygous Males Which Received Fused from the Father.—Such males were mated to fused females and to heterozygous females. In both cases only wild-type progeny were secured, showing again that the paternal gene is not transmitted. From the first type of mating 625 offspring were secured; from the second, 176.

Further Tests of Heterozygous Males Which Received Fused from the Mother.—In addition to the tests of this kind considered above, matings were made with males from fused mothers by wild-type fathers. Such males mated to females of the same constitution gave a total of 45 wild-type females: 60 fused females: 106 wild-type males; 129 fused males. The proportion of the sexes is, of course, not significant, because the females and males came from different mothers.

Conclusion

From the above observations it is concluded that the pair of autosomes, under consideration here, undergoes selective segregation in the male such that only the maternal member is transmitted. This agrees with the evidence from two other species previously considered, and makes it seem probable that such segregation is characteristic of the autosomes in these species.

[1] This investigation has been aided by a grant from the NATIONAL RESEARCH COUNCIL, *Committee for Research in Problems of Sex.*

[2] Metz, C. W., 1926. "An Apparent Case of Monocentric Mitosis in Sciara (Diptera)," *Science*, n. s., **63**, 190–191. Metz, C. W., 1926. "Genetic Evidence of a Selective Segregation of Chromosomes in Sciara (Diptera)," *Proc. Nat. Acad. Sci.*, **12**, 690–692. Metz, Moses, and Hoppe, 1926. "Chromosome Behavior in Sciara (Diptera). I: Chromosome Behavior in the Spermatocyte Divisions," *Zeitsch. induk. Abstammungs- u. Vererb.*, **42**, 237–270. Metz, C. W., 1927. "Chromosome Behavior and Genetic Behavior in Sciara (Diptera). II: Genetic Evidence of Selective Segregation in S. Coprophila," *Ibid.*, **45**, 184–200. Metz, C. W., 1928. "Genetic Evidence of a Selective Segregation of Chromosomes in a Second Species of Sciara (Diptera)," *Proc. Nat. Acad. Sci.*, **14**, 140–141.

[3] Cf. under footnote 2, Metz, Moses and Hoppe, 1926.

[4] Metz and Ullian, 1929. "Genetic Identification of the Sex Chromosomes in Sciara (Diptera)," *Proc. Nat. Acad. Sci.*, **15**, 82–85.

[5] Cf. papers referred to under footnote 2, and also: Moses and Metz, 1928. "Evidence That the Female Is Responsible for the Sex Ratio in Sciara (Diptera)," *Proc. Nat. Acad. Sci.*, **14**, No. 12, 928–932. Metz and Moses, 1928. "Observations on Sex-Ratio Determination in Sciara (Diptera)," *Ibid.*, **14**, No. 12, 928–932.

THE OCCURRENCE OF CHROMOSOME VARIANTS IN NICOTIANA ALATA LK. ET OTTO

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In 1927 Miss Mabel L. Ruttle showed (*Univ. Calif. Publ. Botany*, **11**, pp. 159–176) that 18 somatic chromosomes are characteristic of our cultures of *N. alata* and that a considerable number of pairs exhibit distinctive chromosome morphology. Within this chromosome garniture further investigation has shown that one can distinguish 4 long chromosomes with median constrictions, 12 of intermediate length—of which 4 are satellited, 6 possess median or submedian constrictions and 2 subterminal constrictions—and 2 somewhat shorter chromosomes with subterminal constrictions. Miss Ruttle further found that 8–10 rather than the normal 9–9 II-M PMC occurred in relatively high numbers. It occurred to us that a portion at least of the variation characteristic of our inbred strains of *alata* might therefore be referable to monosomic and trisomic states and studies of chromosome number and behavior in this species have been under way for some time.

In a population of 80 plants from controlled pollination one monosomic *alata* was obtained. From a progeny of normal ♀ x monosomic ♂ 57 plants were examined cytologically and from the reciprocal, 22 plants.

Of the former, 55 plants possessed the normal 18 chromosomes and 2 showed 19. Of the latter, 18 showed 18 while 4 possessed in excess of 18 chromosomes but none of them corresponded to the simple trisomic types found in the former population. The chromosome numbers of these four plants were 22, 25, 26 and 27. Among these 79 plants none was in the $2n-1$ condition. It was possible in most cases to distinguish more or less accurately which chromosome pairs were involved in the plants showing increase in total number of chromosomes.

Judging from our evidence the loss or addition of a single chromosome is reflected in a little if any alteration in external morphology of *alata*. By contrast, in *Tabacum*, at least 4 different mono- and trisomic types are distinguished by clean-cut distinction in plant characters. The addition, in *alata*, of a complete or nearly complete haploid set produced no change in external morphology, while the addition of 7 chromosomes resulted in significant alteration and the addition of 4 produced a highly abnormal plant.

PMC and EMC studies demonstrated that in trisomic *alata* there is a considerable incidence of non-conjunction. Possibly the presence of an unbalanced chromosome set may increase an inherent tendency to this phenomenon characteristic of *alata*. Detailed examination of II stages in PMC of a trisomic revealed the frequent occurrence of diploid metaphases and suspended first anaphases. On the assumption that similar conditions obtain in EMC, the origin of the observed variation in chromosome number in the progenies referred to above, can be explained. Thus the 27-chromosomed individual arose from the union of a 17-chromosomed egg and an $n + 1$ pollen grain. Likewise the 25-chromosomed plant originated from a 17-chromosomed ♀ $\times$ $n - 1$ ♂ and is therefore a doubly modified triploid. The 26-chromosomed plant undoubtedly was produced from the union of a 17-chromosomed egg and a normal 9-chromosomed pollen grain.

It would appear, therefore, that, in *alata*, the presence of more, and presumably also of less, than the normal chromosome garniture results in a considerable incidence of non-conjunction and diploid meioses. There is here a suggestion that tetraploid individuals of *alata* might be produced and also that stable races with chromosome numbers greater than 9_{II} might eventually be obtained. This suggestion is based upon the probability of being able to combine in a soma homologous chromosomes derived from parents in which the same chromosome or chromosomes are duplicated. That, in Nicotiana, four homologous chromosomes will behave as two pairs is shown by the chromosome situation in a tetraploid race of the 16_{II} species *suaveolens* which exhibits 32_{II} at I–M.

By way of summary, we would suggest that *N. alata* would appear to be somewhat valuable material for cytogenetic studies because of the

frequency with which, in it, non-disjunction, non-conjunction, and diploid meioses occur and also because of the rather distinctive character of its somatic chromosome morphology which should often make it possible to determine cytologically the exact chromosome constitution of aberrant individuals which result from these abnormal or at least unorthodox meiotic processes.

AN EOGENE TROPICAL FOREST IN THE PERUVIAN DESERT

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About six miles southeast of Punta Pariñas, Peru—the westernmost extremity of South America—there is a thin outcrop projecting above the salina which contains quantities of silicified fruits and seeds of early Tertiary age. I visited the locality in 1927 and made a large collection, the study of which has just been completed.

In addition to petrified wood, unrecognized fruits and seeds, otoliths and vertebrae of fishes, insect egg cases and fungi, thirty-one species of vascular plants have been identified. Beside 7 species referred to the form genus Carpolithus, there are three species of palm nuts and 21 species of dicotyledons.

The following sixteen plant families, representing thirteen orders, have been recognized: Arecaceae, Moraceae, Myristicaceae, Anonaceae, Leguminosae, Rutaceae, Euphorbiaceae, Humiriaceae, Sapindaceae, Anacardiaceae, Vitaceae, Malvaceae, Dilleniaceae, Boraginaceae, Rubiaceae, and Cucurbitaceae. The genera identified are the following: Palmocarpon, Ficus, Virola, Anona, Leguminosites, Fagara, Jatropha, Saccoglottis, Vantanea, Matayba, Sapindoides, Cupanoides, Anacardium, Ampelocissus, Cissus, Malvacarpus, Tetracera, Lithospermites, Uragoga, Psychotria, Cucurbites, and Carpolithus.

Although studies of carpological remains from relatively recent geological horizons and from localities in temperate latitudes have been cultivated for some time, this is the first account of any considerable assemblage of material of this sort from a tropical environment. The age of this material is either upper Eocene or lower Oligocene.

The major feature of general interest resulting from this study is its bearing on the climate of the early Tertiary of the region. At the present time the coastal strip of Peru is an arid region, usually denominated as "desert." This is due to its location in the trade wind belt and between

lofty mountains on the east and a cold coastal current on the west. Cultivated crops are possible only by irrigation in the valleys of the few through streams from Andean sources.

The fossil fruits and seeds described are sufficiently well distributed among the natural orders to show conclusively that at the time they were living in the region, there was a much greater rainfall than at present, and so distributed through the year as to support a forest cover of tropical vegetation appropriate to the latitude, and similar to that which now characterizes the Choco region of western Colombia, or the wet trans-Andean country of the upper Amazon basin.

The nearest existing relatives of the fossils are, with but two or three exceptions, found in wet tropical situations, either in the region east of the present mountains, or in Central or northern South America. The fossil locality lies so near the present southern border of the tropical rain belt, that it cannot be precisely determined whether this belt was shifted southward in response to the replacement of the cool ocean current from the south by a warm current from the north—such as occurred temporarily in 1925—or what part was taken by the very much less elevated Andean axis of Tertiary time.

That such a shift of currents was not a temporary happening like those which bring abundant rains to the region about once in every generation, of which that of 1925 was the last, is shown by the length of time during which wet conditions persisted in the Tertiary. This at least must have been sufficiently long to bring about forested conditions, since nearly all of the fossils represent arborescent forms.

The additional fact that similar remains as well as petrified tree trunks of large size occur at other geological horizons in the region, as well as the consideration of the improbability of the preservation or the discovery of such a deposit if it represented merely climatic conditions which had endured for but a few centuries, points to the conclusion that wet tropical conditions were extant throughout the major portion of the Tertiary period. This is rendered all the more probable by the fact that several of these fossil species have also been found in Ecuador and as far north as Colombia, there apparently having been no barriers sufficient to prevent the free spread of the plants in either a north and south or in an east and west direction.

ON THE STABILITY OF GASEOUS STELLAR STRUCTURES

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In a recent series of papers[1] Sir James Jeans has developed a very interesting new theory of stellar structures and stellar evolution. He has found that gaseous stars are unstable so that the existence of stars implies substantial deviations from the gas laws. These stability considerations lead him to another important conclusion—that the rate of liberation of stellar energy is approximately independent of temperature and density. All these results are somewhat unexpected, and have caused some bewilderment in the scientific world, raising doubt as to the basis of theoretical astrophysics, which, in the hands of different investigators, can lead to results that are dissimilar and even mutually exclusive. It therefore seems to be of importance to analyze Jeans' stability criterion somewhat further, and to see if all stars, independently of their masses, can be rescued from Jeans' "cast iron" law, or if at least some important statistical groups can be saved.

The following calculations show that if the changes in ionization of a star in a perturbed state are taken into account, not only dwarfs and giants, but even supergiants are stable in the gaseous state. Certain very infrequent stars of exceedingly large mass may be, at least partly, in a "liquid" state; this category contains chiefly stars of Class O and early B stars.

In what follows we employ Jeans' elegant method of treating the problem, introducing additional terms depending on the energy of ionization, which was neglected by him.

1. *Stability Criterion, Taking into Account Changes in Ionization.*—The criterion is to be deduced from the differential equations of hydrodynamical motion and of perturbed radiative equilibrium. Neglecting terms depending on viscosity, they may be written

$$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{d}{dr}\left(p_G + \frac{1}{3}aT^4\right) - k^2\frac{M_r}{r^2} \qquad (1)$$

$$d\frac{E}{\rho} + p\,d\frac{1}{\rho} = G - \left(\frac{T}{T_0}\right)^{7+n}\left(\frac{r}{r_0}\right)^7 G_0. \qquad (2)$$

We use here Jeans' notation,[2] except that in the second equation E denotes the energy per unit volume. The expression $p_G + 1/3\,aT^4$ denotes the total pressure p; M_r is the mass within a sphere of radius r; and

ρ, T, r, and G are the perturbed values of the density, temperature, distance from the center, and generation of energy per unit mass, equilibrium values being indicated by the subscript 0. Now for E, the energy per unit volume, we have

$$E = \frac{3}{2}\frac{RT\rho}{m_H\mu} + aT^4 + I$$

representing the sum of the kinetic energy of the particles, the energy of the imprisoned radiation, and the total ionization energy I per unit volume. The molecular weight, μ, is expressed in terms of that of the hydrogen atom, and the mass of the hydrogen atom is m_H.

When studying the effects of change in ionization on the stability, we certainly cannot regard μ as constant. If $\left(d\frac{E}{\rho}/dt\right)_0$ is the value of $d\frac{E}{\rho}/dt$ calculated on the hypothesis of constant ionization, we have

$$d\frac{E}{\rho}/dt = \left(d\frac{E}{\rho}/dt\right)_0 + \frac{3}{2}\frac{RT}{m_H}\,d\frac{1}{\mu}/dt + d\frac{I}{\rho}/dt = \left(d\frac{E}{\rho}/dt\right)_0 + S\frac{dT}{dt}$$

where $S = \frac{3RT}{2m_H}\,d\frac{1}{\mu}/dT + d\frac{I}{\rho}/dT$. Following Jeans we consider the star affected by a disturbance which is uniformly spread throughout the stellar interior. We suppose, moreover, that the star is affected by a uniform expansion or contraction. This kind of change is rather unfavorable to stability, because viscosity will not here exert any braking action. We denote by λ the ratio of gas pressure p_G to radiation pressure, and adopt for the rate of generation of energy the expression $G \propto \rho^\alpha T^\beta$. The opacity coefficient is taken as proportional to $\rho T^{-(3+s)}$, and to allow for possible deviations from the gas laws we adopt $p_G \propto T\rho^{(1+s)}$ and $p_G = (1 + \xi)NRT$, where N and R have their usual meanings for the theory of gases.

After putting $T = T_0 + \delta T$ and $r = r_0 + \delta r$, the next procedure, following Jeans, is as below. We eliminate δT and its derivatives from (1) and (2), and take the time factor to be of the form $e^{\nu t}$. Substituting this factor in the resulting third order differential equation for δr we obtain the following characteristic equation of a given stability problem[3]

$$A\nu^3 + B\nu^2 + C\nu + D = 0.$$

The necessary and sufficient conditions for stability are

$$\frac{BC}{A} > D > 0.$$

The first protects the star from overstability, the second from unstable expansion or contraction. The coefficients A to D are:

$$A = \frac{R}{m\mu}\left(\frac{3}{2} + \frac{12(1+\xi)}{\lambda} + \frac{m_H\mu S}{R}\right)$$

$$B = \frac{G_0}{T_0}(7 + n - \beta)$$

$$C = \frac{k^2 M_r R}{r_0^3 \mu m}\left[\frac{\lambda+4}{\lambda+1}\left(\frac{3}{2} - \frac{m_H\mu S}{R} + 3\xi\right) + \frac{3s\lambda}{\lambda+1}\left(\frac{3}{2} + \frac{m_H\mu S}{R} + \frac{12(1+\xi)}{\lambda}\right)\right]$$

$$D = G_0 \frac{k^2 M r}{r_0^3 T_0}\left[\frac{\lambda+4}{\lambda+1}(3\alpha + \beta - n) + \frac{3s\lambda}{\lambda+1}(7 + n - \beta)\right].$$

Jeans supposed the ionization energy to be in a constant ratio to the energy of the material particles, this ratio being incorporated into γ, the ratio of specific heats; he finally adopted the value $\gamma = 5/3$, thus neglecting ionization energy altogether. We differ in analyzing the different kinds of energy separately, so that γ does not enter explicitly into our formula. Putting $S = 0$ we get Jeans' expression for A, B, C, and D if $\gamma = 5/3$ and the coefficient C is corrected according to Vogt.[4]

After some algebraical transformations the stability criterion takes the form

$$\beta + 3\alpha - n < \left(\frac{7}{3} + \alpha\right)\left[3 - \frac{3}{2(1+\xi)} - \frac{m_H\mu S}{R(1+\xi)}\right]\frac{\lambda}{\lambda+4} \qquad (3)$$

and

$$\beta + 3\alpha - n > -\left(\frac{7}{3} + \alpha\right)\frac{9s\lambda}{\lambda + 4 - 3s\lambda}. \qquad (4)$$

From these two inequalities (with $S = 0$) Jeans derived his important conclusions. It is in fact not difficult to see that in this case, for a gaseous star, where $s = \xi = 0$, condition (3) gives values of $\beta + 3\alpha$ that are too small, even for stars of moderate mass.

We now investigate the effect of S. Let x_0 denote the relative number of bared nuclei, and x_i the number of nuclei with only i bound electrons. Let ψ_i be the energy corresponding to the i level, while ψ_0 is zero. The values of ψ are the same for all electrons belonging to the same ring. Supposing that the stellar matter is composed of one element only, we obtain for the total ionization energy per unit volume

$$I = \frac{\rho}{m}\Sigma x_i \Psi_i; \qquad \Psi_0 = 0$$

where Ψ_i is the total energy of all bound electrons up to the i level inclusive; it is a simple function of the ionization potentials from various levels (ψ_i). The symbol m denotes the atomic weight in C.G.S. units. On the other hand

$$m_{H\mu} = \frac{m}{I + \Sigma(Z - i)x_i}$$

where Z is the atomic number. We therefore have

$$S = \frac{3}{2}\frac{RT}{m}\frac{d\Sigma x_i(Z - i)}{dT} + \frac{1}{m}\frac{d\Sigma x_i\Psi_i}{dT} = -\frac{3}{2}\frac{RT}{m}\frac{d\Sigma x_i i}{dT} + \frac{1}{m}\frac{d\Sigma x_i\Psi_i}{dT}.$$

Ionization is determined by the following equations:

$$\sum_0^Z x_i = 1$$

$$\text{Log}\frac{x_{i-1}}{x_i} = -\frac{69000\,Z^2}{T(1+\tau)^2} - \frac{3}{2}\log T + 14.118 - \log\lambda + \log\sigma_i. \quad (5)$$

This is the Saha-Fowler formula, the density being eliminated by the ratio of gas pressure to radiation pressure.[5] The symmetry number is denoted by σ_i and the ordinal number of the quantum rings by τ, where τ is zero for the first ring. Our notation for the quantum levels differs from the usual one, which counts the i's from the lowest level.

As the stars are very condensed toward their centers, we should expect that the central portions of the stellar masses would give us a fair sample of stellar material; we shall therefore consider only the central temperatures. A calculation of S in the general case is very complicated, and we shall therefore analyze only the relatively simple condition where the ionization is fairly complete, and only the K ring is to be taken into account; we can scarcely doubt that the central parts of the stars are very near to this ideal condition. Within a star built according to Eddington's model, the elements lighter than titanium retain no electrons, and those lighter than tin ($Z = 50$) are bared at least down to the K ring. For Jeans' theoretical values of the central temperature, and moderate molecular weights (if these can in some way be reconciled), the ionization is even larger. In what follows we disregard the L ring and all higher rings.

With the assumptions just stated we have for the stellar interior

$$x_0 + x_1 + x_2 = 1; \qquad \frac{x_0}{x_1} = X; \qquad \frac{x_1}{x_2} = 2X,$$

where $2X$ is the antilogarithm of the right side of (5) when τ is put equal to zero; the ratio x_1/x_2 is deduced by taking into account that for the K ring $\sigma = 2$. In this manner we get

$$x_1 = \frac{2X}{2X^2 + 2X + 1}; \qquad x_2 = \frac{1}{2X^2 + 2X + 1}.$$

On the other hand, the term of the stability criterion involving S is, in the particular case under consideration,

$$\frac{\mu m_H}{m}\left[\frac{\psi_1}{R} - \frac{3}{2}T\right]\left[\frac{dx_1}{dT} + \frac{2dx_2}{dT}\right].$$

Now

$$\frac{dx_1}{dT} + \frac{2dx_2}{dT} = -\frac{(4X^2 + 8X + 2)}{(2X^2 + 2X + 1)^2}\frac{dX}{dT} = -\frac{(4X^2 + 8X + 2)X}{(2X^2 + 2X + 1)^2}\left(\frac{\psi_1}{kT^2} - \frac{3}{2T}\right)$$

and thus the term under examination turns out to be

$$\frac{-m_H\mu S}{R} = +\frac{\mu m_H 2X}{m}\frac{(2X^2 + 4X + 1)}{(2X^2 + 2X + 1)^2}\left(\frac{\psi_1}{kT} - \frac{3}{2}\right)^2$$

where

$$\frac{\psi_1}{kT} = \frac{69000\,Z^2}{T \times \text{Mod.}}.$$

We see that the terms expressing the variation of ionization contribute favorably to the stellar stability, the contribution being always in the direction of increasing the stability. To calculate its amount we must investigate special stellar models for giants and supergiants.

2. *Stability of Eddington's Model.*—The best data are those for Capella. Taking a molecular weight 2.1, corresponding to titanium ($Z = 22$), $\lambda = 2.5$, $T = 9 \times 10^6$, according to Eddington,[6] we obtain $X = 0.38$, and $\frac{\psi_1}{kT} = 8.5$; the inequality (3) is therefore $\beta + 3\alpha < 2.8 + \alpha$ if $n = 1/2$, according to Kramers' theory. To understand this result we must recall that if the generation of energy is occasioned by collisions of any type, $3\alpha + \beta \geq 3.5$, and $\alpha = 1$; if the proximity of other bodies is operating, $3\alpha + \beta \geq 3.0$, and $\alpha \geq 1$. We see, therefore, that Capella built according to Eddington's model is certainly stable for such mechanisms of energy generation. This is an important matter, for Capella ($M = 4.2\ \odot$) is a typical giant. On the other hand, it is of interest that if the variation in ionization is not taken into account, Capella is not stable, for in this case $3\alpha + \beta < 1.8 + 0.6\alpha$, which is rather small for a gaseous star.

Let us analyze the stability of a typical supergiant, δ Cephei ($M = 9\ \odot$). If Eddington's data are adopted,[7] $T = 6.2 \times 10^6$, $\lambda = 1.2$, $Z = 22$, and δ Cephei as a gaseous star seems to be unstable, for $3\alpha + \beta$ is less than $1.4 + 0.4\alpha$. But the data in this case are very uncertain. Taking a slightly different value of 8×10^6 for the central temperature, we get $\beta + 3\alpha < 2.8 + \alpha$. We therefore conclude that supergiants built according to Eddington's model are stable in a gaseous state.

Not so favorable is the case of stars of very great mass. A calculation for V Puppis ($M = 19\ \odot$) gives a value for $3\alpha + \beta$ well below what is permissible for a gaseous star. Because of the small values of λ and X, these stars, if gaseous bodies, are certainly unstable. Here, then, we are forced to admit deviations from the gaseous state. But these "liquid" stars, however interesting they are, cannot represent a statistically important step in stellar evolution; being confined to Class O and the early B classes, they represent a very rare phenomenon.

3. *Stability of Jeans' Gaseous Model.*—Here the question is somewhat complicated. It is known that for Jeans' model the solution for Z^2/m, where Z is the atomic number and m the atomic weight, diverges seriously,[8] thus suggesting that the gaseous model is wholly impossible. If on the other hand we tentatively explain the divergence by too low a value for the coefficient F in the opacity law as used by Jeans, we may use the values of λ and T given by Jeans' theory, because even big errors in the assumed value of F lead only to relatively small errors in T.

If we calculate the right side of inequality (3), assuming Jeans' data for Capella ($\mu = 2.5$), we get values that are too low, indicating that this star, if gaseous, cannot be stable. But taking $\mu = 2.2$ (iron) we calculate[9] $\lambda_c = 3.1$, $T_c = 13 \times 10^6$, and finally $3\alpha + \beta < 2.8 + \alpha$, as for Eddington's model. Thus an ordinary giant built of iron according to this model is certainly stable. The escape of a gaseous supergiant from instability seems to be much narrower. If made of titanium ($\mu = 2.1$), δ Cephei has $T_c = 8 \times 10^6$, and $\lambda_c = 1.5$. With these data, $3\alpha + \beta < 2.2 + 0.7\alpha$, and δ Cephei is stable for the second mode of generation of energy. A small change in the adopted data would bring it above the margin of instability for the the first mode of generation of energy also. It is hardly necessary to add that stars of exceedingly large mass built according to the Jeans model are quite unstable if they are gaseous, so that they escape instability only by dint of their deviations from the gaseous state.

Our results may be summarized as follows. The change in ionization during the perturbed motion of a gaseous star contributes favorably to the stability. The contribution is not large, but it is sufficient to insure the stability of typical giants and supergiants, and gaseous dwarfs are stable even without taking into account the variations of μ and I. There is a distinct indication that stars of exceedingly large mass cannot be gaseous. But in this point we do not adopt Jeans' line of argument;[10] he assumed that Pierce's massive star is very near to the gaseous state, deduced for it a very small value of s and ξ, and afterward on the basis of this result, concluded that deviations from the gaseous state are necessary, whatever the mass of the star.

[1] *Mon. Not., R. Astr. Soc.*, **87**, 1927 (400); **88**, 1928 (393).

[2] *Ibid.*, **87**, 1927 (401), equations 1, 2–6.
[3] *Ibid.*, **87**, 1927 (405), equation 9.
[4] *Astr. Nach.*, **232**, 1928 (1).
[5] *Astronomy and Cosmogony*, 1928 (147).
[6] *The Internal Constitution of the Stars*, 1926 (145).
[7] *Ibid.*, 1926 (147).
[8] *Astronomy and Cosmogony*, 1928 (101).
[9] *Ibid.*, 1928 (96).
[10] *Ibid.*, 1928 (120,141).

ON THE SPHERICALLY SYMMETRICAL STATICAL FIELD IN EINSTEIN'S UNIFIED THEORY OF ELECTRICITY AND GRAVITATION

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In recent papers Einstein[1] has developed a theory of electricity and gravitation based on a fundamental asymmetrical tensor ${}^{a}h_{\lambda}$ which serves to associate with each point of a continuum X_4 a quadruple of axes ("4-Bein") and consequently determines a parallelism of directions valid for the whole of space and not only for the infinitesimal neighborhood of a point. The index a of ${}^{a}h_{\lambda}$ is associated with the directions of the axes at each point, while the index λ is an ordinary vector index. The difference between the index λ and the index a is signalized by the facts, first, that "a" indices can be contracted only with "a" indices and λ indices only with λ indices; second, that there is no distinction between co- and contravariant "a" indices, so that any two such indices can be contracted. Contravariant λ indices are introduced in the familiar manner by taking ${}_{a}h^{\lambda}$, the normalized minors of the determinant $\left| {}^{a}h_{\lambda} \right|$. The coefficients $g_{\lambda\mu}$ of Riemann geometry are defined by $g_{\lambda\mu} = {}^{a}h_{\lambda}\,{}^{a}h_{\mu}$.

There are certain important tensors of the new Einstein geometry into which the identification of the co- and contravariant "a" indices does not enter. Chief among these is the tensor $\Lambda^{\nu}_{\mu\sigma}$ which is defined as

$$\Lambda^{\nu}_{\mu\sigma} = \frac{1}{2}\left({}_{a}h^{\nu}\frac{\partial {}^{a}h_{\mu}}{\partial x^{\sigma}} - {}_{a}h^{\nu}\frac{\partial {}^{a}h_{\sigma}}{\partial x^{\mu}} \right).$$

The expression which Einstein interprets as the vector electromagnetic potential φ_{μ} is $\Lambda^{\sigma}_{\mu\sigma}$. The purpose of this note is to investigate the nature of this potential under the assumptions made in the previous (1916)

Einstein theory to obtain Schwarzschild's spherically symmetrical statical field.

In the present theory the differential element of length along the direction a is ${}^{a}h_{\lambda}\,dx^{\lambda}$. Let the principal directions which we consider be those of the three spherical coördinates r, φ, θ and of the time t. By considerations of spherical symmetry similar to those used in Schwarzschild's solution[2] the antisymmetric tensor ${}^{a}h_{\lambda}$ must be a function of r only. The elements of length along the a directions ($a = 1, 2, 3, 4$) of the quadruple of axes are, therefore, with the simplest choice of the ${}^{a}h_{\lambda}$:

$$\begin{aligned} d^1x &= U(r)dr + M(r)dt \\ d^2x &= {}^2h_2 r \sin\theta\, d\varphi \qquad ({}^2h_2 = 1) \\ d^3x &= {}^3h_3 r d\theta \qquad ({}^3h_3 = 1) \\ d^4x &= N(r)dr + W(r)dt. \end{aligned}$$

If we follow Eddington in imposing the additional condition of the symmetry of time as far as past and future are concerned[3] we must put $M = N = 0$. We thus obtain the following table of ${}^{a}h_{\lambda}$:

$$\begin{array}{llll} {}^1h_1 = U(r), & {}^1h_2 = 0, & {}^1h_3 = 0, & {}^1h_4 = M(r), \\ {}^2h_1 = 0, & {}^2h_2 = 1, & {}^2h_3 = 0, & {}^2h_4 = 0, \\ {}^3h_1 = 0, & {}^3h_2 = 0, & {}^3h_3 = 1, & {}^3h_4 = 0, \\ {}^4h_1 = N(r), & {}^4h_2 = 0, & {}^4h_3 = 0, & {}^4h_4 = W(r). \end{array}$$

The corresponding ${}_{a}h^{\lambda}$ are:

$$\begin{array}{llll} {}_1h^1 = \dfrac{W}{UW - NM}, & {}_1h^2 = 0, & {}_1h^3 = 0, & {}_1h^4 = \dfrac{N}{WU - NM}, \\ {}_2h^1 = 0, & {}_2h^2 = 1, & {}_2h^3 = 0, & {}_2h^4 = 0, \\ {}_3h^1 = 0, & {}_3h^2 = 0, & {}_3h^3 = 1, & {}_3h^4 = 0, \\ {}_4h^1 = \dfrac{M}{UW - NM}, & {}_4h^2 = 0, & {}_4h^3 = 0, & {}_4h^4 = \dfrac{U}{UW - NM}. \end{array}$$

We now have, since $g_{\lambda\mu}dx^{\lambda}dx^{\mu} = \sum(dk_x)^2$, the following table for the coefficients $g_{\lambda\mu}$:

$$\begin{array}{llll} g_{11} = U^2 + N^2, & g_{12} = 0, & g_{13} = 0, & g_{14} = UM + WN, \\ g_{21} = 0, & g_{22} = r^2 \sin^2\theta, & g_{23} = 0, & g_{24} = 0, \\ g_{31} = 0, & g_{32} = 0, & g_{33} = r^2, & g_{34} = 0, \\ g_{41} = MU + WN, & g_{42} = 0, & g_{43} = 0, & g_{44} = M^2 + W^2. \end{array}$$

It follows from this that the Einstein definition of the $g_{\lambda\mu}$, expressed in invariantive form, becomes

$$g_{\lambda\mu} = {}_{ab}\gamma\ {}^{a}h_{\lambda}\ {}^{b}h_{\mu}$$

where the ${}_{ab}\gamma$ are the coefficients of the Euclidean quadratic differential form in four dimensions expressed in arbitrary coördinates, in this case polar.

The contravariant $g^{\lambda\mu}$ are given in the following table:

$$g^{11} = \frac{M^2 + W^2}{(N^2 + W^2)(U^2 + M^2) - (MU + NW)^2}, \qquad g^{12} = 0, \qquad g^{13} = 0,$$

$$g^{14} = \frac{WN + MU}{(N^2 + W^2)(U^2 + M^2) - (MU + NW)^2}, \qquad g^{21} = 0,$$

$$g^{22} = \frac{1}{r^2 \sin^2\theta}, \qquad g^{23} = 0, \qquad g^{24} = 0, \qquad g^{31} = 0, \qquad g^{32} = 0,$$

$$g^{33} = \frac{1}{r^2}, \quad g^{34} = 0, \quad g^{41} = \frac{UM + WN}{(N^2 + W^2)(U^2 + M^2) - (MU + NW)^2},$$

$$g^{42} = 0, \quad g^{43} = 0, \quad g^{44} = \frac{U^2 + N^2}{(N^2 + W^2)(U^2 + M^2) - (MU + NW)^2}.$$

The only non-vanishing $\Lambda^{\nu}_{\mu\sigma}$ are readily seen to be the following:

$$\Lambda^{4}_{41} = -\Lambda^{4}_{14} = \frac{1}{2}\frac{UW' + NM'}{UW - NM}$$

$$\Lambda^{1}_{41} = -\Lambda^{1}_{14} = \frac{1}{2}\frac{WM' + MW'}{UW - NM}.$$

As a consequence the Einstein electromagnetic potential vector φ_{μ} has the following components:

$$\varphi_1 = \Lambda^{\mu}_{1\mu} = -\frac{1}{2}\frac{UW' + NM'}{UW - NM}, \qquad \varphi_2 = \varphi_3 = 0,$$

$$\varphi_4 = \Lambda^{\mu}_{4\mu} = \frac{1}{2}\frac{WM' + MW'}{UW - NM}.$$

In the case where we have symmetry of past and future $M = N = 0$, and the potential vector reduces to

$$\varphi_1 = -\frac{1}{2}\frac{\partial}{\partial r}\log W, \qquad \varphi_2 = \varphi_3 = \varphi_4 = 0.$$

In fact, for this it is only necessary that $M = 0$. We shall presently show

that the Einstein field equations require that in this case W = const., and that therefore $\varphi_1 = 0$. Thus, *an electromagnetic field is incompatible in the new Einstein theory with the assumptions of static spherical symmetry and of symmetry of past and future.* Indeed, even if $W' \neq 0$, the electromagnetic *field* (not the *potential*) vanishes.

The expression $g^{\mu\nu}\Lambda^{\alpha}_{\mu\beta}\Lambda^{\beta}_{\nu\alpha}$ is one of the fundamental invariants of the new Einstein theory. In this case this invariant has the value

$$H = \frac{(M^2+W^2)(UW'+NM')^2-2(WN+MU)(WM'+MW')(UW'+NM'+(U^2+N^2)(WM'+MW')^2}{4[(M^2+W^2)(U^2+N^2)-(MU+NW)^2](UW-NM)^2}$$

Einstein's field equations are obtained by equating to zero the variation of the integral over space-time of the corresponding invariant density. In our case this variational equation becomes

$$\delta \int_0^{\infty} hr^2 H dr = 0 \qquad (h = |\,{}^{a}h_{\lambda}\,|).$$

In the case of time symmetry in past and future this becomes

$$\delta \int_0^{\infty} r^2 \frac{W'^2}{UW}\, dr = 0$$

and leads to the result $W' = 0$, or W = const.[4] Thus there is no bond between space and time. Further under the given assumptions all the components of the tensor $\Lambda^{\nu}_{\mu\sigma}$ vanish, and the continuum is Euclidean as Einstein himself has shown (first paper, loc. cit., p. 220). Therefore *the gravitational field also vanishes.*

It is interesting to notice that the conditions which lead to Schwarzschild's solution in Einstein's 1916 theory lead here to a situation which seems to be devoid of physical significance. Inasmuch as all the experimental confirmations of general relativity are essentially tests of the Schwarzschild solution, the new Einstein theory lacks at present all experimental confirmation.

[1] A. Einstein, "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus," *Sitzungsberichte der preussischen Akademie der Wissenschaften*, **1928**, pp. 217–221; "Neue Moglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität," *Ibid.*, pp. 224–227; "Zur einheitlichen Feldtheorie," *Ibid.*, **1929**, pp. 2–7.

[2] A. S. Eddington, *The Mathematical Theory of Relativity*, p. 83, Cambridge, **1923**.

[3] It has been shown by G. D. Birkhoff, *Relativity and Modern Physics*, p. 253, that Schwarzschild's solution can be obtained without this last assumption.

[4] Since writing this paper the authors have learned from Dr. H. Müntz that the new Einstein field equations of the 1929 paper do not yield the vanishing of the gravitational field in the case of spherical symmetry and time symmetry. In this case he has been able to obtain results checking the observed perihelion of mercury.

UNIMOLECULAR REACTIONS

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This paper is devoted in the main to an exposition of a view of unimolecular reactions based on new quantum theory considerations. The fundamental ideas involved are very similar to those made use of recently in work on thermionic emission[1] and, more closely, the problem of radioactive disintegration.[2] In fact, the seeming necessity for a complex structure in the case of atomic and unimolecular disruption makes a formal analogy at least natural.

It is conceivable that a molecule like N_2O_5 may decompose in several ways and yield end products differing in excitation, charge or composition—let us fix attention on a particular mode of decomposition described by the severing of a certain bond. We have in mind, then, two linked atomic complexes which roughly speaking may be characterized by a representative particle of reduced mass $\mu = \frac{m_1 m_2}{M}$ (where M, m_1, m_2 are the masses of N_2O_5 and the end products, respectively) at a distance γ from the force center. The manifold inter-relations between the various submolecular systems lend credibility to our fundamental postulate that for the bond singled out the potential energy function is of the form illustrated—the essential feature is, of course, the assumption of the intermediate region in which the energy exceeds the dissociation energy W_0. As is known, in contradistinction to classical mechanics, the new theories allow transition of the representative point from region A to region B without increase in the vibratory energy above W. It is seen that such transition implies (assuming of course $W > W_0$) dissociation.

In a sense the predictions attending the acceptance of the above mechanism and those derived from the Lindemann theory[3] on which most of the more plausible recent theoretical work has been based will be of the same character—however, the picture here is altogether different for there is no difference in type between an activated and a reactive molecule and the time lag of change from the one to the other is interpreted as a measure merely of the probability that the representative point will escape from region A. Our proposed explanation meets two tests satisfactorily—in the first place in contrast with, for instance, radiation theories, it indicates that only the more complicated molecules, where extraordinary couplings may be expected, can exhibit the phenomenon of unimolecular reaction and it accounts also in a natural way for constancy of the reaction rate under widely varying conditions; for instance, even solution in liquids.[3]

As an extreme simplification let us characterize the molecule's motion, except for the exponential time factor, by the Schrödinger equation

$$\frac{d^2\psi}{dr^2} + l^2(W - V_{(r)})\psi = 0, \tag{1}$$

where

$$l^2 = 8\pi^2\mu/h^2.$$

It may be shown as a consequence of some results of Jeffries[4] that a fair approximation to the probability that the representative point traversing the region A_1^2 from left to right will penetrate to B is[5]

$$\gamma(W) \sim \frac{8((W - W_0)W)^{1/2}}{(W_1 + W_0)W_1 \cosh 2l \int_{x_1}^{x_2} (V_{(x)} - W)^{1/2}\, dx - W(W - W_0)}. \tag{2}$$

For a general case we might expand the integral in equation (1) in powers of $W_1 - (W - W_0)$. We are unable to take into account the undoubtedly marked dependence of $V_{(x)}$ on the specifications of other bonds. In the interest of directness of exposition we assume the approximation

$$V_{(x)} = W_0 + 2W_1\left(\frac{x}{x_0}\right) - W_1\left(\frac{x_1}{x_0}\right)^2 \qquad a < x < b \tag{3}$$

where $x = r - r_1$.

Hence, as is easily verified,

$$2l \int_{x_1}^{x_2} (V_{(x)} - W)^{1/2} dx = B(W_1 + W_0 - W) \tag{3.1}$$

with

$$B = 2\pi^2 x_0 (2\mu/W_1)^{1/2}. \tag{3.2}$$

The probability of dissociation/unit time is therefore

$$P(W) = \nu\, \gamma(W), \tag{4}$$

where ν is the vibration frequency given as to order of magnitude by

$$2((W - W_2)/\mu)^{1/2}/a.$$

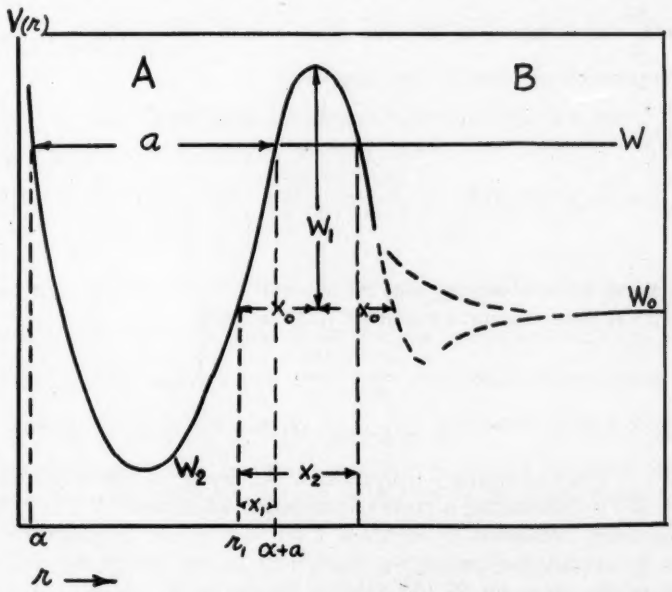
Consistency obliges us to assume that, except at extraordinarily high pressures or temperatures, every molecule given energy above $W_0 + W_1$ in the special bond/unit time dissociates. This number R may be evaluated indirectly by appeal to the principle of Detailed Balancing according to which R is also given by the number of molecules which would have left the energy interval $W_0 + W_1 \longrightarrow \infty$ by collision at equilibrium, assuming the molecules had been stable.

Thus
$$R = 4\sigma^2 \left(\frac{\pi kT}{M}\right)^{1/2} N \int_{W_0 + W_1}^{\infty} N(W) f(W) dW \qquad (5)$$

with the ordinary notation of kinetic theory where $f(W)$ is the fraction of collisions effective in changing the energy content of the bond. Assuming this fraction to be of the order of unity there results

$$R = 4\sigma^2 \left(\frac{\pi kT}{M}\right)^{1/2} N^2 (\text{exp.} - (W_0 + W_1)/kT).$$

The estimate of $N(W)$ used throughout this work considers the kinetic and potential energy of vibration, only, as being important. Several



papers have been written recently based primarily on the possibility of different forms for $N(W)$ depending on the number of degrees of freedom expected to be important.[6] Apart from this, however, there seems to be the possibility of a different kind of modification. The assumption is invariably made that the energy is expressible as a positive definite quadratic form in the coördinates and their conjugated momenta. Yet, for the problem of molecular dissociation such an assumption may be wide of the truth in the energy range of interest. It does not seem satisfactory to correct for this by adding one or two higher power terms—the finiteness of the energy for dissociation requires inverse powers. In view of

the fact that one of the outstanding difficulties in the explanation of the unimolecular reaction rate of N_2O_5 is the failure of ordinary distribution laws to give computed activation rates of proper magnitude, this remark may be of importance and it is the writer's intention to study the modifications in partition function entailed.

The contribution to the decomposition rate for the range $W_0 \longrightarrow W_0 + W_1$ is

$$\int_{W_0}^{W_0 + W_1} P(W)\, \overline{N(W)}\, dW \tag{7}$$

where $\overline{N(W)}$ is the number of molecules of energy W. On equating to 0 the algebraic sum of the molecules added or removed from the region $W \pm \frac{dW}{2}$ by collision or reaction, there results (on making use, again, of the principle of Detailed Balancing)

$$4\sigma^2(\pi kT/M)^{1/2}N[\overline{N(W)} - N(W)] + \overline{N(W)}P(W) = 0. \tag{8}$$

Thus

$$P(W)\, \overline{N(W)} = \frac{P(W)\, N(W)}{1 + P(W)/4\sigma^2 \left(\frac{\pi kT}{M}\right)^{1/2} N}. \tag{7.1}$$

It is worth while observing that for $W = W_0 + W_1$, the rigorous value of $P(W)$ is close to ν and accordingly (7.1) becomes

$$\frac{\alpha\nu N\, N(W)}{\alpha N + \nu}, \tag{7.2}$$

where $\alpha = 4\sigma^2(\pi kT/M)^{1/2}$.

Since $\nu > > \alpha N$ at ordinary temperatures and pressures, this is effectively $\alpha N\, N(W)$ substantiating a previous statement (cf. supra).

The exact evaluation of equation 7 (cf. 7.1) would be awkward—in order to exhibit the qualitative predictions of the theory we slur the effect of the range for W immediately inferior to $W_0 + W_1$. Accordingly, the denominator variation is neglected and the integral written as

$$16 \int_{W_0}^{W_0 + W_1} \nu[(W - W_0)(W)]^{1/2}(W_0 + W_1 - W)(\exp - B(W_0 + W_1 - W))\, N(W)\, dW. \tag{9}$$

On writing $W_2 \sim 0$

$$W \sim W_0 \exp W/W_1$$

$$W_0 + W_1 - W \sim (W_0 + W_1) \exp - W_1/W_0 + W_1$$

and introducing the abbreviations

$$F = \frac{16}{4\sigma^2}\left(\frac{2M}{\mu\pi kT}\right)^{1/2} W_0(W_0 + W_1)N$$

$$\gamma = \frac{1}{kT} - B - \frac{W_1}{W_0(W_0 + W_1)},$$

9 may be integrated as

$$\frac{F}{\gamma^{3/2}} e^{-BW_1 - \frac{W_0}{W_1 + W_0} - \frac{W_0}{kT}}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(W_1\gamma)^{1/2} - \frac{(W_1\gamma)^{1/2}}{2} e^{-W_1\gamma}\right] \quad (10)$$

Experimental data indicate that for $T \sim 350°\text{K.}$; $\frac{W_0}{kT} \sim 40$. If W_1 is not too small a fraction of W_0, we may approximate the first term in the bracket of (10) by

$$\frac{\sqrt{\pi}}{4} - \frac{e^{-W_1\gamma}}{2W_1\gamma}.$$

Our result for the velocity of reaction may be shown to be

$$K = \frac{F}{\gamma^{3/2}} e^{-\frac{W_0}{kT}}\left[\frac{\sqrt{\pi}}{2} e^{-BW_1 - \frac{W_0}{W_0 + W_1}} - e^{-\frac{W_1}{kT}}\left[\frac{1 + W_1\gamma}{(W_1\gamma)^{1/2}} e^{-\frac{W_1^2}{W_0(W_1 + W_0)}}\right] + \alpha e^{-\frac{W_0 + W_1}{kT}}\right. \quad (11)$$

In order for B (cf. eq. (3.2)) to be small in comparison with $\frac{1}{kT}$, $T \sim 350°$ K., it is necessary that $x_0 < 10^{-10}$ cm. or so, which is not unplausible. On writing $\exp - W_0/W_1 + W_0 \sim \frac{1}{4}$, etc., and positing also that the reacting molecules with energies above $W_0 + W_1$ are relatively few in number, equation (11) reduces still further to an Arrhenius type of formula, namely,

$$K \sim 4\nu_0 W_0^{1/2}(W_0 + W_1)N(kT)^{3/2} e^{-\frac{W_0}{kT} - BW_1}.$$

In principle at least the parameters may be determined from band-spectrum data as well as from reaction rate experiments.

The considerations of this paper provide a means for estimating the equilibrium point of the unimolecular decomposition, for, in the reverse reaction, we have the problem of the representative point passing the barrier in the direction $B \longrightarrow A$. The probability involved is simply related to $\gamma(W)$ and hence we should have to compare the number of collisions for which such a transition would be possible with the frequency of vibration of the completed molecule. The matter will not, however, be pursued here.

[1] Fowler and Nordheim, *Proc. Roy. Soc.*, **A119**, 1 (1928).

[2] R. W. Gurney and E. U. Condon, *Phys. Rev.*, **33**, 127 (1929); Gamow, *Zeit. Phys.*, **51**, 204 (1928).

[3] Cf. C. N. Hinshelwood, *Chem. Rev.*, **3**, 227 (1926).

[4] Jeffries, *Lond. Math. Soc.*, p. 428, **1923**.

[5] Although the expression used for $P(W)$ is valid provided that W has one of the characteristic values, the results will not be far wrong if we employ the same form for a continuously varying W.

[6] O. K. Rice and H. C. Ramsperger, *J. A. C. S.*, **49**, 1617 (1927); O. K. Rice, *Proc. Nat. Acad. Sci.*, **14**, 113 (1928); N. S. Kassel, *Ibid.*, **14**, 23 (1928); R. H. Fowler, *Statistical Mechanics*.

A GENERALIZATION OF ROLLE'S THEOREM WITH APPLICATION TO ENTIRE FUNCTIONS

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1. Rolle's theorem for a function of a real variable $f(x)$ gives a sufficient condition for the existence of roots of the derivative in a certain interval, namely, between two consecutive zeros of $f(x)$. The object of the present note is to find a similar criterion for the existence and number of the roots of the derivative of a function of a complex variable $f(z)$, in the circle (C) drawn on the segment joining two zeros of $f(z)$ as a diameter, provided there is no other zero of $f(z)$ on or inside that circle.

2. We may assume, by a convenient choice of the origin, that the zeros of $f(z)$ are the points $\pm \lambda$ on the real axis. Further, we suppose that $f(z)$ is holomorphic in a region (D) containing (C). We have therefore

$$f(z) = (z^2 - \lambda^2)g(z),$$

$g(z)$ being holomorphic in (D) and without zeros in (C). The zeros of $f'(z)$ are given by the equation

$$(z^2 - \lambda^2)\frac{g'(z)}{g(z)} + 2z = 0. \qquad (1)$$

Let $g'(z)/g(z) = 1/\lambda u(z)$ and solve (1) as a quadratic equation in z; we have

$$z = \lambda\,[\pm(1 + u^2)^{1/2} - u] = F(z).$$

The number of roots of this equation in the circle (C) is given by Cauchy's integral

$$n = \frac{1}{2\pi i}\int_C \frac{F'(z) - 1}{F(z) - z}\,dz = \frac{1}{2\pi}\{\arg[F(z) - z]\}_C.$$

Let us consider first the simple case when the two determinations of $F(z)$ are uniform in (C), and, moreover, the region (Γ) corresponding to (C) in the transformation $Z = F(z) - F(z)$ being one of the determinations—is inside (C). It follows immediately that

$$\{\arg[F(z) - z]\}_C = 2\pi$$

and therefore $$n = 1.$$

From the assumptions made on $F(z)$ it is very easy to obtain the corresponding conditions for $g(z)$, and we get finally the following result:

THEOREM 1.—*If, in the transformation* $Z = X + iY = g'(z)/g(z)$, *the circle* (C) *corresponds to a region* (Δ) *such that the relations* $X = 0, |Y| > 1/\lambda$ *are never simultaneously verified at a point of* (Δ), *there exists one and only one zero of* $f'(z)$ *in the circle* (C).

In other words, the region (Δ'), corresponding to (C) in the transformation $u = u(z)$, is contained in the plane of the variable u, cut by the segment L joining the points $u = +i$ and $u = -i$.

3. Let us consider next the case when the boundary of (Δ') cuts L at two points M and N, the segment MN being interior to (Δ'), and suppose, moreover, that the correspondence between (C) and (Δ), and therefore, between (C) and (Δ') is a (1, 1) correspondence. Let P and Q be the points of the circumference (C) corresponding to M and N, respectively, and PlQ the curve corresponding to MN.

Now the transformation

$$U = \lambda[(1 + u^2)^{1/2} - u]$$

with the proper determination (namely, $(1 + u^2)^{1/2} > 0$, when u is a real positive number), carries the u-plane cut by the segment L into the interior of (C); the right hand border of the cut L corresponding to the semi-circle of (C) to the right of the imaginary axis, and the left border of L to the other semi-circle of (C).

By dividing (Δ') into two regions (Δ_1') and (Δ_2') by means of MN, we thus manage to transform (C) into two regions (Γ_1) and (Γ_2) situated inside (C), by the transformation $Z = F(z)$, where the determination of $F(z)$ has been chosen as said above. These two regions have as parts of their boundary corresponding to MN, two arcs m_1n_1 and m_2n_2 of the circle (C) symmetrical with respect to the imaginary axis; if M' and N' are the points where m_1m_2 and n_1n_2 cut that axis, we have

$$\Im\{M'\} = -\lambda\,\Im\{M\}$$
$$\Im\{N'\} = -\lambda\Im\{N\}$$

The number of zeros of $f'(z)$ inside (C) is then given by

$$n = \frac{1}{2\pi i}\left[\int_{C_1} \frac{F'(z) - 1}{F(z) - z}\, dz + \int_{C_2} \frac{F'(z) - 1}{F(z) - z}\, dz\right],$$

(C_1) and (C_2) being the boundaries of the parts of (C) which correspond respectively to (Γ_1) and (Γ_2). We can always assume that $\mathfrak{I}\{M'\} < \mathfrak{I}\{N'\}$. A study of the above integrals leads to the following results:

$$\begin{aligned}
n = 2 \quad & \text{if} \quad \mathfrak{I}\{P\} > \mathfrak{I}\{M'\} \quad \text{and} \quad \mathfrak{I}\{Q\} < \mathfrak{I}\{N'\} \\
n = 1 \quad & \text{if} \quad \mathfrak{I}\{P\} > \mathfrak{I}\{M'\} \quad \text{and} \quad \mathfrak{I}\{Q\} > \mathfrak{I}\{N'\} \\
\text{or} \quad & \text{if} \quad \mathfrak{I}\{P\} < \mathfrak{I}\{M'\} \quad \text{and} \quad \mathfrak{I}\{Q\} < \mathfrak{I}\{N'\} \\
n = 0 \quad & \text{if} \quad \mathfrak{I}\{P\} < \mathfrak{I}\{M'\} \quad \text{and} \quad \mathfrak{I}\{Q\} > \mathfrak{I}\{N'\}
\end{aligned}$$

which can be expressed in the form $n = 1 + \delta$, where

$$2\delta = \frac{\mathfrak{I}\{P\} - \mathfrak{I}\{M'\}}{|\mathfrak{I}\{P\} - \mathfrak{I}\{M'\}|} - \frac{\mathfrak{I}\{Q\} - \mathfrak{I}\{N'\}}{|\mathfrak{I}\{Q\} - \mathfrak{I}\{N'\}|}. \qquad (2)$$

We will refer to δ as the index of the cut PQ. A similar study may be carried out when the boundary of (Δ') cuts L in a single point, one of the points $\pm i$ being interior to (Δ').

4. If we consider finally the general case when the boundary of (Δ') cuts L in any number of points, the correspondence between (C) and (Δ') being (1,1), we can divide (Δ') by a suitable number of auxiliary cuts, into a finite number of regions, which either do not cut L, or are of one of the two kinds studied above. This gives, therefore, the following theorem.

THEOREM II.—*If the correspondence $Z = g'(z)/g(z)$ between the circle (C) and the region (Δ) is (1,1), the number of zeros of $f'(z)$ inside the circle (C) is given by*

$$n = 1 + \sum_k \delta_k, \qquad (3)$$

δ_k being the index of the k^{th} cut of (C), each δ_k being given by a formula of the type (2).

5. The restriction imposed upon the transformation $Z = g'(z)/g(z)$ can be very easily removed; the transformation $u = u(z)$ defines a certain Riemann surface, and the region (Δ') spreads over a finite number of sheets. If we draw a cut joining the points $+i$ and $-i$ in each one of these sheets, we can carry out the argument as above. There can be only one exception, when one of the points $\pm i$ is a branch-point of the surface; then several cuts will merge into a single one on the surface, which changes slightly the formula (2) giving the indices. The value of those indices can easily be found in each particular case, according to the way in which the sheets are connected at the branch-point.

6. The criterion we have thus obtained enables us to prove Rolle's theorem for analytic functions of a real variable. For the cuts of the circle (C) are then symmetrical to each other with respect to the real axis, and as no one can be symmetrical to itself, owing to the fact that $f(z)$ has no other zero in (C), their number is even. On the other hand, two symmetrical cuts have the same index. Therefore, n is an odd number, which proves that there is at least one zero of $f'(z)$ on the real axis between the points $\pm\lambda$.

7. Another immediate application is to Laguerre's theorem for entire functions of genus zero and one, of the type

$$f(z) = e^{kz} \Pi \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}, \tag{4}$$

where k and a_n are real numbers such that $\sum 1/a_n^2$ converges. Applying theorem II to the circle (C_n) with the segment $(a_n a_{n+1})$ as a diameter, it is shown very easily from (2) that each cut of the circle has the index zero. Therefore, there is one and only one zero of $f'(z)$ in (C_n), and it is, of course, on the real axis.

This method can be extended to the function

$$F(z) = e^{\alpha i z} f(z)$$

where $f(z)$ is given by (4) and α is a real number. If $|\alpha| < 1/\lambda_n$ where $2\lambda_n = |a_{n+1}| - |a_n|$, there is still one and only one zero of $F'(z)$ in the circle (C_n).

8. It is possible to extend Laguerre's theorem to functions of genus zero with a distribution of zeros of a much greater generality.

Let

$$f(z) = \Pi \left(1 - \frac{z}{a_n}\right)$$

where a_n are complex numbers, with modulus increasing with n, such that $\sum 1/|a_n|$ converges; further, we suppose that no two zeros are equal.

Let

$$|a_n| = n^{\sigma(n)}$$

where $\sigma(x)$ can be chosen as a twice differentiable function of x; further, let

$$\varphi(x) = \frac{d}{dx}[\sigma(x) \log x] = \frac{\sigma(x)}{x} + \sigma'(x)\log x > 0.$$

As the function is of genus zero, we have

$$\lim_{x \to \infty} \sigma(x) > 1.$$

We are going to consider separately two classes of functions of genus zero:

I. Let us suppose first that $x^{1/2}\varphi(x)$ tends to $+\infty$ with x, and, further, that $x\varphi(x)$ is an increasing function for sufficiently large values of x.

Let us assume the following hypotheses on the distribution of zeros in the plane:

A. If $\alpha_n = \left(\overrightarrow{Oa_n}, \overrightarrow{a_n a_{n+1}}\right)$, $|\alpha_n| \leqslant \alpha_0 < \frac{\pi}{2}$,

α_0 being a fixed angle.

B. Let D and D' be the straight lines drawn at right angles with the segment $a_n a_{n+1}$ from the points a_n and a_{n+1}; there are at most N zeros of $f(z)$ between D and D', N being independent of n.

C. There is no zero of $f(z)$ in the circle (C_n) with $a_n a_{n+1}$ as a diameter.

Then, there is one and only one zero of $f'(z)$ in each circle (C_n) for $n > n_0$, n_0 being a sufficiently large number. Further, if b_n is that zero, $|a_{n+1} - b_n| / |a_{n+1} - a_n|$ tends to zero with $1/n$.

The proof follows from theorem I, which we apply by showing that in the circle (C_n), $X > 0$, when $n > n_0$.

II. Suppose that $x^{1/2}\varphi(x)$ tends to zero with $1/x$, and that, either $x\varphi(x)$ is an increasing function of x, tending to $+\infty$ with x (in the case of functions $f(z)$ of order zero), or $1 < a \leqslant x\,\varphi(x) \leqslant a'$, where a and a' are fixed numbers (in the case of functions $f(z)$ of positive order $\rho < 1$).

Further, let us assume that:

A. $$\varphi\left(x + \frac{A}{\varphi(x)}\right) \geqslant B\,\varphi(x) \qquad \begin{bmatrix} 0 < B \leqslant 1 \\ A > 0 \end{bmatrix}$$

whatever the constant A may be, for all $x > x_0$; (x_0 and B may depend on A.

B. $$\varphi\left(x \pm \frac{\gamma}{\varphi(x)}\right) \geqslant h\,\varphi(x)$$

where $0 < \gamma \leq c$ and $x > x_0$, c being a fixed number, and $h > h_0$ for functions of order zero, $h > h_0^{1/2}$ for functions of positive order. Here h_0 is the root of the equation

$$\psi\left(1 + \frac{1}{h_0}\right) - \psi(1) = 2h_0$$

where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The value of h_0 is $h_0 = 0.79\ldots$

We then make the following hypotheses on the distribution of the zeros:

C. $|\alpha_n| \leqslant \alpha_0 < \pi/2$, α_n having the same meaning as above.

D. Let (S_n) and (S'_n) be the circles tangent to the segment $a_n a_{n+1}$

at its middle point A_n, and having a common radius

$$R_n = Kn\varphi(n).OA_n,$$

K being a conveniently chosen number. We assume that all the zeros of $f(z)$ are outside these circles.

E. There are no zeros of $f(z)$ in the circle (C_n).

Then there is one, and only one zero of $f'(z)$ in the circle (C_n) for $n > n_0$, n_0 being a sufficiently large number.

The proof follows again from theorem I, by showing that in the circle (C_n), $|Y| < 1/\lambda_n$ when $n > n_0$.

It must be noted in this case that there is a necessary geometrical relation between α_n and R_n; it is easy to see that

$$\sin|\alpha_n| < \frac{1}{Kn\varphi(n)}.$$

For the functions of order zero such that $a < x^{1/2}\varphi(x) < b$, where a and b are fixed numbers, neither of the above methods applies; it is then necessary to make further hypotheses on the zeros.

DUALITY RELATIONS IN TOPOLOGY

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In some recent papers[1] I have introduced the relative cycles for point sets and the associated relative boundary relations and homologies which may be of four types: absolute, modular, relative, relative modular. Various considerations lead one also to introduce a couple of invariants analogous to the Betti-numbers and for all these I have given loc. cit. proofs of some very general relations, in particular of duality which include all those previously known.

In going over the whole question I have recently had occasion to revise the proofs and extend the results somewhat. The extensions are along the line of much information concerning the relative torsion coefficients which occur, however, only when the subset G of the carrying complex is polyhedral. I do not wish to dwell on these here. The modified proofs are noteworthy and the changes shall now be indicated in outline. Their object was to extend as far as possible Poincaré's own proof for the duality relations of an M_n without boundary and to avoid wherever possible

Kronecker indices as in the original proofs. The two basic elements in Poincaré's proof are the incidence matrices of the cells and the construction of a dual complex.

Let C_n be a complex of Veblen's type which defines an M_n and let C_n^* be its dual. The cells of C_n^* can be so oriented with respect to those of C_n, that the incidence matrix of the h and $(h-1)$-cells of C_n is the transverse of the similar matrix for the dimensions $n-h+1$ and $n-h$ of C_n^*. From this Poincaré's duality theorems follow and likewise Veblen's and Alexander's extensions to the modular cases. This is manifestly as straightforward and direct a procedure as could be desired, and now for its generalization.

Let us call *regular* a cell of C_n which fulfills the same requirements as if C_n defined an M_n without boundary. Let G be a subset of C_n. If every cell of C_n not on G is regular we shall say that C_n is a *manifold relatively to* G, or more briefly, that $C_n - G$ is a *manifold*.

Take first the case where G is a subcomplex of C_n. Let C_n' be the first derived complex of C_n (regular subdivision of C_n). The sum of the cells of C_n' that have a vertex on the regular cell E_h of C_n but do not meet E_h, is an $(n-h)$-cell E_{n-h}^*, the *transverse* of E_h. The sum of these transverses is a complex C^*, the dual of C_n relative G. Among its properties the following are of particular interest: (*a*) $C - C^* = N$ neighborhood of G on C sum of the cells of C_n' with a vertex on G. (*b*) The cells of C^* whose sum is the boundary of N are the transverses of those of $C_n - G$ with a vertex on G.

The comparison between the incidence matrices of $C_n - G$ and C^* yields all the duality relations corresponding to G, a subcomplex of C_n. More generally of course G may be assumed merely to be a polyhedral complex on C_n.

Suppose now that *G is an arbitrary closed set on C_n*, with $C_n - G$ a manifold. Consider a subdivision C_n' of C_n and let N be the sum of all its cells whose closure meets G. It is a neighborhood of G of the same type as above. Subdivide further the cells of N alone as far as desired and let N' be the analogous neighborhood constructed by means of the new subdivision, and so on. The totality of all the cells of $C_n - N$, $N - N'$, etc., is a denumerable set of regular cells which constitutes what may be described as an *infinite* complex K_n, defining an *infinite* manifold. The transverses of the cells of K_n constitute another infinite complex K^*, the *dual* of the first. The incidence matrices for these complexes are exactly as for ordinary (finite) complexes with a finite number of non-zero elements in each row or column, but the number of rows or columns is infinite (denumerable).

The machinery is thus at hand for extending Poincaré's scheme of things with few modifications relatively. In particular we can avoid the seem-

ingly involved and unnatural Vietoris cycles whose place is now taken by the infinite cycles (eventually fractionary) composed of cells of K or K^*.

Infinite complexes are susceptible of other applications. They are notably convenient in proving the invariance of the relative Betti and torsion numbers. They have already been considered for $n = 2$ by Kerekjarto,[2] but there is no hint of the above applications in his work.

[1] These PROCEEDINGS, **13** (1927), 614–622, 805–807; *Ann. of Math.*, (2) **29**, (1928), 232–254.

[2] *Vorlesungen über Topologie.*

GROUPS WHICH ADMIT THREE-FOURTHS AUTOMORPHISMS

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A group G is said to admit three-fourths automorphisms if it is possible to establish at least one (1,1) correspondence between its operators in which exactly three-fourths of these operators correspond to their inverses. This is the only condition imposed on G in the present article. It is easy to prove that whenever G admits at least one such automorphism it must admit exactly three distinct ones, and it must contain exactly three abelian subgroups of index 2. In each of these three automorphisms all the operators of two of these subgroups correspond to their inverses while the remaining operators correspond to their inverses multiplied by the commutator of order 2 contained in G. The continued product of these three automorphisms is an automorphism of G in which each of the operators of the central corresponds to its inverse while every other operator corresponds to its inverse multiplied by the commutator of order 2 contained in G. Hence G must also admit such an automorphism whenever it admits a three-fourths automorphism, and this automorphism is invariant under the group of automorphisms of G while a three-fourths automorphism is not necessarily invariant under this group.

A necessary and sufficient condition that a group admits a three-fourths automorphism is that its central is of index 4, and hence its order must be divisible by 8. If its order is not a power of 2 then G must be the direct product of an abelian group of odd order and a non-abelian group of order 2^m, and every such direct product admits three-fourths automorphisms whenever its Sylow subgroup of order 2^m has this property. At most three-fourths of the operators of a non-abelian group can correspond to their inverses in an automorphism of the group, and hence the

category of the groups under consideration is composed of the totality of the non-abelian groups which separately admit an automorphism in which the maximal relative number of the operators of such a group may be made to correspond to their inverses in an automorphism. The quaternion group and the octic group constitute the most elementary examples of such groups.

A necessary and sufficient condition that a three-fourths automorphism of G is the identity automorphism is that only one-fourth of the operators of G have orders which exceed 2. Hence this is a characteristic property of the catagory of groups composed of the direct product of the octic group and any group which involves no operator whose order exceeds 2.[1] The other two three-fourths automorphisms of each of the groups of this category are inner automorphisms of order 2 and are conjugate under the group of isomorphisms of G. The three three-fourths automorphisms of G generate an abelian group whose order is either 4 or 8. If this order is 4, then each is an inner automorphism while none of them is an inner automorphism when this order is 8. The product of two distinct three-fourths automorphisms is always an inner automorphism of order 2, and each of the three inner automorphisms of this order is the product of two three-fourths automorphisms of G. The Hamiltonian groups of order 2^m are characterized by the fact that their three three-fourths automorphisms are also their three inner automorphisms of order 2. The same category of groups is also characterized by the fact that the continued product of its three three-fourths automorphisms is the identity.

A group is said to admit an α-automorphism whenever each of its operators corresponds to its α^{th} power in one of the possible automorphisms of the group. It is known that the α-automorphisms are the only invariant automorphisms of an abelian group and that they are also invariant under the group of automorphisms of a non-abelian group but they do not necessarily constitute all the invariant operators of the latter group. A group can obviously not admit an α-automorphism unless α is prime to its order and hence it is only necessary to consider the additional coditions which a group must satisfy in order that it may admit an α-automorphism. Suppose that all the commutators of a given group are invariant and have orders which do not exceed 2. It results at once that if s_1 and s_2 are two operators of such a group then we can establish an α-automorphism between its operators whenever $\alpha = 1 \bmod 4$; for if s_1 and s_2 correspond respectively to s_1^α and s_2^α then it results that $s_1 s_2$ corresponds to $(s_1 s_2)^\alpha$ since $(s_1 s_2)^\alpha = s_1^\alpha s_2^\alpha$ when the given conditions are satisfied. Hence we have established the following theorem: *If all the commutators of a group are in its central and have orders which do not exceed* 2 *then this group admits an* α*-automorphism whenever* α *is both prime to its order and also congruent to* 1 *mod* 4, *and only then.* From this theorem

it follows directly that a group which admits three-fourths automorphisms must also admit an α-automorphism whenever α is of the form $1 + 4k$ and is prime to the order of the group, and only then.

If a group G which admits three-fourths automorphisms admits also an α inner automorphism of order 2, its Sylow subgroup of order 2^m must also have this property. One of the abelian subgroups of index 2 contained in this Sylow subgroup must therefore be composed of all of its operators which are not of highest order, and its commutator of order 2 must be generated by one of its operators of highest order. Moreover, this highest order must exceed 4. When these conditions are satisfied any non-invariant operator of this Sylow subgroup which is not of highest order must transform every operator of this subgroup into the same power of itself, and hence it must transform these operators according to an α inner automorphism of order 2. This furnishes a proof of the following theorem:

Necessary and sufficient conditions that a group which admits three-fourths automorphisms admits also an α inner automorphism of order 2 are that its Sylow subgroup of order 2^m involves an abelian subgroup of index 2 composed of all its operators which are not of highest order and that the fourth power of an operator of highest order generates its commutator of order 2.

If a three-fourths automorphism of G is also an α-automorphism it must be the identity automorphism since an operator of order 2^β, $\beta > 1$, must correspond to its inverse in each of the other three-fourths automorphisms. As the continued product of the three three-fourths automorphisms of G can also not be an α-automorphism unless it is the identity automorphism it results from the theorem noted at the close of the preceding paragraph that the group formed by the α-automorphisms of G cannot have more than two operators in common with the group generated by its three-fourths automorphisms, and if these two groups have two operators in common the latter must be of order 8. The order of the group formed by the α-automorphisms of G is equal to the number of the distinct natural numbers of the form $1 + 4k$ which are prime to the order of G and do not exceed the order of the largest operator contained in G. This group appears in the central of the group of automorphisms of G but it cannot contain more than half the operators of this central when the continued product of the three three-fourths automorphisms of G is of order 2 since this continued product is also in this central, as was noted above.

When G is extended by an operator which transforms it according to a three-fourths automorphism and has its square in G there results a group whose order is twice the order of G. To determine all the groups which can be constructed in this way when G is given it may be noted that the only operators besides the identity which are invariant under such a

group are the operators of order 2 which appear in the central of G. It therefore results from the general theory for constructing all the groups which contain invariantly a given group of prime index[2] that if the multiplier s_0 is not the identity it can be selected in as many different ways as there are different complete sets of conjugates operators of order 2 under the group of isomorphisms of G in the central of G. Hence the following theorem: *The number of distinct groups which may be formed by extending a given group G by means of an operator which transforms it according to a three-fourths automorphism and has its square in G is one more than the number of the complete sets of conjugate operators of order 2 under the group of isomorphisms of G which appear in the central of G.*

[1] G. A. Miller, *Ann. Math.*, **7**, 55, 1906.

[2] G. A. Miller, these PROCEEDINGS, **14**, 819, 1928.

THE RANK FUNCTION OF ANY SIMPLE ALGEBRA

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1. *Introduction.*—No special properties of the rank function of a simple algebra have been given in the literature except for the limiting cases of a total matric algebra and a normal division algebra. We consider here the general simple algebra, first represented as a normal algebra over its central field, the field of all of its elements commutative with every one of its elements, and then as a simple algebra over F, and obtain the rank and irreducibility of the rank function.

2. *Properties of an Algebra A over $F(v)$ with Respect to F.*—Let F be any infinite field and v be an element whose minimum equation with respect to F is irreducible with respect to F and has degree a. Then $F(v)$ is an algebraic field of order a over F. Consider any algebra A over $F(v)$ which is a linear associative algebra with a modulus (denoted by 1) and has order b with respect to $F(v)$. Then, if $u_0 = 1$, and $u_0, u_2, \ldots, u_{b-1}$ are a basis of A with respect to $F(v)$, the elements

$$w_{a\beta+\alpha} \equiv v^{\alpha}u_{\beta} \quad (\alpha = 0, 1, \ldots, a-1;\ \beta = 0, 1, \ldots, b-1)$$

are linearly independent with respect to F and form a basis of A considered as an algebra over F. Hence *A has order $a\,b$ with respect to F.*

We may write the general element of A to be $x = \sum_{\gamma=1}^{ab} \xi_{\gamma}w_{\gamma}$ where $\xi_1, \ldots,$ ξ_{γ} are independent variables in F. Then a known theorem[1] states that

x is a root of an uniquely defined rank equation $(R\omega; \xi_1, \ldots, \xi_{ab}) = 0$ of least degree ρ, the rank of A with respect to F. The coefficient of the highest power ω^ρ of ω in $R(\omega)$ is unity and all further coefficients are polynomials in $\xi_1, \ldots, \xi_{ab}$ with coefficients in F. However, we may write $\eta_\beta = \sum_{\alpha=0}^{a-1} \xi_{a\beta+\alpha} v^\alpha$ and express x by means of the formula $x = \sum_{\beta=0}^{b-1} \eta_\beta u_\beta$. Applying our known theorem we find that x is a root of a uniquely defined equation $S(\omega; \eta_0, \eta_1, \ldots, \eta_{b-1}) = 0$, of degree r, the rank of A with respect to $F(v)$. The coefficient of ω^r in $S(\omega)$ is one and all further coefficients are polynomials in $\eta_0, \eta_1, \ldots, \eta_{b-1}$ with coefficients in $F(v)$. Let $K = F(\xi_1, \xi_2, \ldots, \xi_{ab})$, a function field. Then the equation

$$S(x; \eta_0, \eta_1, \ldots, \eta_{b-1}) \equiv T(x; \xi_1, \xi_2, \ldots \xi_{ab}; v) = 0$$

expresses x^r as a polynomial in x of degree $\leq r - 1$ and with coefficients in $K(v)$.

Consider the linear set E composed of all polynomials in x with coefficients in $K(v)$. Every element of E is expressible in the form $\sum_{\alpha=1}^{a} \sum_{\beta=1}^{r} \epsilon_{\alpha\beta} v^{\alpha-1} x^{\beta-1}$ with $\epsilon_{\alpha\beta}$ in K. Therefore the order of E, considered as a linear set over K, is less than or equal to ar. But every power of x is in E and consequently there are at most ar such linearly independent powers. Therefore the $ar + 1$ powers $x^\delta (\delta = 0, 1, \ldots, ar)$ are linearly dependent with respect to K so that there exist $ar + 1$ elements $t_0, \ldots, t_{ar}$ of K such that

$$t_0 x^0 + t_1 x + \ldots + t_{ar} x^{ar} = 0.$$

But every element of K is a rational function with coefficients in F of $\xi_1, \ldots, \xi_{ab}$. Let the greatest common denominator of $t_0, t_1, \ldots, t_{ar}$ be q so that $q\, t_i = q_i \ (i = 0, 1, \ldots, ar)$, where q_i are polynomials in $\xi_1, \ldots, \xi_{ar}$ with coefficients in F. Thus

$$\lambda(\omega) \equiv q_{ar}\omega^{ar} + q_{ar-1}\omega^{ar-1} + \ldots + q_0 = 0$$

has x as a root and $\lambda(\omega)$ is known (loc. cit.) to be divisible by $R(\omega; \xi_1, \ldots, \xi_{ar})$ the rank equation of A with respect to F. Hence, $\rho \leq ar$ and we have

LEMMA 1. Let A be a linear associative algebra with a modulus over $F(v)$ of order a. Let ρ be the rank of A considered as over F and r be the rank of A with respect to $F(v)$. Then $\rho \leq ar$.

We shall next assume that the rank equation of A with respect to $F(v)$ is irreducible. Then by a theorem of Hilbert[2] there exists a set of elements $\eta_0', \eta_1', \ldots, \eta_{b-1}'$ of $F(v)$ such that $S(\omega; \eta_0', \ldots, \eta_{b-1}')$ is irreducible with respect to $F(v)$ and is therefore the minimum equation of $x' = \sum_{\beta=1}^{b-1} \eta_\beta' u_\beta$ with respect to $F(v)$. The *field* $F(v, x')$ has order ar and

as a basis the elements $v^{\alpha-1}(x')^{\delta-1}$ ($\alpha = 1, 2, \ldots, a$; $\delta = 1, 2, \ldots, r$) and by the theory of algebraic fields contains an element y of grade ar with respect to F. But y is in A and has grade less than or equal to ρ, the rank of A. Hence $\rho \geq ar$. But by lemma 1 we know that $\rho \leq ar$. Thus $\rho = ar$. Also if $y = \sum_{\gamma=1}^{ab} \xi'_\gamma w_\gamma$ then $R(\omega; \xi'_1, \ldots, \xi_{ab})$ has degree ar and leading coefficient unity and is the minimum equation of y, of grade r. Since y is an element of a field its minimum equation is irreducible. Hence $R(\omega; \xi'_1, \xi'_2, \ldots, \xi'_{ab})$ is irreducible and consequently $R(\omega; \xi_1, \xi_2, \ldots, \xi_{ab})$ in which $\xi_1, \xi_2, \ldots, \xi_{ab}$ are arbitrary parameters of F, is irreducible with respect to $F(\xi_1, \xi_2, \ldots, \xi_{ab})$. Thus we have

THEOREM 1. Let A be a linear associative algebra, with a modulus, and expressible as an algebra over $F(v)$ a field of order a. Let the order of A with respect to $F(v)$ be b and the rank equation of A with respect to (Fv) be irreducible in the field of its coefficients and of degree r. Then the order of A with respect to F is ab and the rank equation of A with respect to F is irreducible in the field of its coefficients and has degree ar.

3. *The Rank of Any Simple Algebra.*—Let A be any simple algebra over F. Then A is known to be expressible as the direct product of a total matric algebra M and a division algebra B, in symbols,

$$A = B \otimes M.$$

Moreover, it is known[3] that if also $A = B_1 \otimes M_1$ where B_1 is a division algebra and M_1 is a total matric algebra then B_1 is equivalent to B and M_1 is equivalent to M. The order of A is the product of the orders of B and M.

Every division algebra is expressible as a normal division algebra over the field of all of its elements commutative with every one of its elements. Call this latter field the *central field* of the algebra. Then if B is a division algebra over F and the central field of B is generated by v and has order s B is a normal division algebra of order t^2 over $F(v)$ of order s. Then the order of B with respect to F is t^2s. Also since B is a normal division algebra of order t^2 over $F(v)$, it has rank t with respect to $F(v)$ and its rank equation with respect to $F(v)$ is irreducible. Then by theorem 1, we deduce

THEOREM 2. Let B be a division algebra of order h over F. Then B is expressible as a normal division algebra of order t^2 over $F(v)$ of order s, its central field, and $h = t^2s$. The rank of B with respect to F is ts and the rank equation of B with respect to F is irreducible in the field of its coefficients.

Next consider $A = B \otimes M$, where M is an r-rowed total matric algebra over F. It is known that by means of the adjunction of a finite number of scalars to $F(v)$, the algebra B' over the resulting field G, which has the

same basis and multiplication constants as B, is equivalent, by means of a transformation of this basis, to the algebra of all t-rowed matrices in G. But the direct product of a t-rowed total matric algebra and an r-rowed total matric algebra is equivalent to a tr-rowed matric algebra. Let $q = t^2r^2$ and $u_1, u_2, \ldots, u_q$ be a basis of $A = B \otimes M$. Then the algebra $A' = B' \otimes M'$ which has the same basis $u_1, \ldots, u_q$ and the same multiplication constants as A, but whose general element is $x = \sum_{\alpha=1}^{q} \xi'_\alpha u_\alpha$ with $\xi'_1, \ldots, \xi'_q$ ranging over G, is a total tr-rowed matric algebra. The rank function of A' is then $R(\omega; \xi'_1, \ldots, \xi'_q)$ a polynomial in ω with leading coefficient unity, further coefficients in $K' = G(\xi'_1, \ldots, \xi'_q)$ and having degree tr.

Let now $\delta(\omega; \xi_1, \ldots, \xi_q)$ be the first characteristic determinant of A. This since A' has the same basis and multiplication table as A the characteristic equation of A' is $\delta(\omega; \xi'_1, \ldots, \xi'_q)$. But it is known that

$$\delta(\omega; \xi'_1, \ldots, \xi'_q) \equiv [R(\omega; \xi'_1, \ldots, \xi'_q)]^{tr}.$$

Thus we have

$$\delta(\omega; \xi_1, \ldots, \xi_q) \equiv [R(\omega; \xi_1, \ldots, \xi_q)]^{tr}.$$

But $R(\omega; \xi'_1, \ldots, \xi'_q)$ is irreducible in $K' \equiv G(\xi'_1, \ldots, \xi'_q)$ and moreover has the symmetric group[4] with respect to this field. Also $R(\omega; \xi_1, \ldots, \xi_q)$ is merely interpretationally distinct from $R(\omega; \xi'_1, \ldots, \xi'_q)$ so that $R(\omega; \xi_1, \xi_2, \ldots, \xi_q)$ is irreducible in K' and has the symmetric group with respect to K'. Let $K = F(v; \xi_1, \ldots, \xi_q)$. Then $\delta(\omega; \xi_1, \xi_2, \ldots, \xi_q)$ is a polynomial in ω with leading coefficient unity and further coefficients in K.

It is easily shown that, since $\delta = R^{tr}$, $R(\omega; \xi_1, \ldots, \xi_q)$ has leading coefficient unity and further coefficients polynomials in $\xi_1, \ldots, \xi_q$ with coefficients in $R(v)$. The equation $R(\omega; \xi_1, \ldots, \xi_q) = 0$ is satisfied by the general element $x = \sum_{\alpha=1}^{q} \xi_\alpha u_\alpha$ of A, where $\xi_1, \ldots, \xi_q$ range over $F(v)$. But then $R(\omega; \xi_1, \ldots, \xi_q)$ is divisible by the rank equation of A with respect to $F(v)$. However, $R(\omega; \xi_1, \ldots, \xi_q)$ is irreducible with respect to K' and consequently with respect to K, a sub-field of K' but which contains the coefficients of $R(\omega; \xi_1, \ldots, \xi_q)$. Hence $R(\omega; \xi_1, \ldots, \xi_q)$ is the rank function of A. We then obtain

THEOREM 3. Let $A = B \otimes M$ be a simple algebra over $H = F(v)$, where B is a normal division algebra of order t^2 over H and M is a total matric algebra of order r^2 over H. Then by the adjunction of a finite number of scalars to H, the algebra A' with the same basis and constants of multiplication as A but over the resulting field H' is equivalent to the algebra of all tr-rowed matrices in H'. Let $q = t^2r^2$. Then the rank function of A is $R(\omega; \xi_1, \ldots, \xi_q)$, the rank function of A', but with

$\xi_1, \ldots, \xi_q$ ranging over H. Hence A has rank tr with respect to H and its rank equation has the symmetric group with respect to $K = H(\xi_1, \ldots, \xi_q)$ and is irreducible in K.

By applying theorem 1 we have

THEOREM 4. Let A be a simple algebra over F expressed by $A = B \otimes M$ where M is a total matric algebra of order r^2 and B is a normal division algebra in t^2 units over its central field $R(v)$ of order s. Then the rank of A is str and the rank function of A with respect to F is irreducible in the field of its coefficients.

[1] Dickson, L. E., *Algebren und ihre Zahlentheorie*, p. 260.

[2] Cf. Noether, E., *Mathematische Annalen*, vol. **78.**

[3] Wedderburn, *Bulletin of the American Mathematical Society*, **31** (1925), 11–13.

[4] Cf. the author's paper, "On the Group of the Rank Equation, etc.," these PROCEEDINGS **14,** (1928), No. 12, pp. 906–7.

CONFORMAL GEOMETRY OF GENERALIZED METRIC SPACES

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1. Let V_n be an n-dimensional generalized metric space—one to which there is assigned an absolute scalar differential invariant $f(x, dx)$, f being positively homogeneous of degree two in $dx^1, \ldots, dx^n$. If $f_{.i} \equiv \partial f/\partial dx^i$ and $g_{ij} \equiv {}^1/_2 f_{.i.j}$ the length of a vector ξ relative to the element dx is by definition

$$\xi^2 = g_{ij}(x,dx)\xi^i\xi^j, \tag{1.1}$$

the summation convention for every repeated index being used.

I define two metrics $f(x, dx)$ and $f'(x, dx)$ as *conformal* if the length of an arbitrary vector in the one is proportional to the length of this vector in the other. Let $\varphi(x, dx)$ be the factor of proportionality; then from (1.1) we obtain

$$g'_{ij} = \varphi g_{ij}$$

and therefore

$$g'_{ij.k} = \varphi g_{ij.k} + \varphi_{.k} g_{ij} = \varphi g_{jk.i} + \varphi_{.i} g_{jk}.$$

Hence

$$\delta^i_j \varphi_{.k} = \delta^i_k \varphi_{.j},$$

from which we get by contraction $\varphi_{.k} \equiv 0$. Hence *the factor of proportionality is at most a point function.*

For convenience we write

$$g'_{ij} = e^{2\sigma} g_{ij}; \tag{1.2}$$

then if g^{ij} is the normalized cofactor of g_{ij} in $|\, g_{\alpha\beta} \,|$, which is assumed not to vanish, we have

$$g'^{ij} = e^{-2\sigma} g^{ij}. \tag{1.3}$$

Let Γ and Γ' be the fundamental affine connections of the two spaces, their components being given[1] by expressions of the form

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \left\{ {i \atop ik} \right\}_{.j} dx^\alpha + \left\{ {i \atop j\beta} \right\}_{.k} dx^\beta + \tfrac{1}{2} \left\{ {i \atop \alpha\beta} \right\}_{.j.k} dx^\alpha dx^\beta \; ; \tag{1.4}$$

$\left\{ {i \atop jk} \right\}$ being the Christoffel symbols of the second kind formed out of the g's. When these components are compared we find that

$$\Gamma'^i_{jk} = \Gamma^i_{jk} - A^{i\alpha}_{.j.k}\, \sigma_\alpha \tag{1.5}$$

$$\text{where } \sigma_\alpha = \partial\sigma/\partial x^\alpha \text{ and } A^{i\alpha} = \tfrac{1}{2} f g^{i\alpha} - dx^i dx^\alpha.$$

If K^i_{jkl} are the components of the affine curvature tensor, we find by means of (1.5) that

$$K'^i_{jkl} = K^i_{jkl} - (A^{i\alpha}_{.j.k}\, \sigma_{,\alpha,l} - A^{i\alpha}_{.j.l}\, \sigma_{,\alpha,k}) + \sigma_{,\alpha}\sigma_{,\beta}\, (A^{h\alpha}_{.k} A^{i\beta}_{.h.l} - A^{h\alpha}_{.l} A^{i\beta}_{.h.k})_{.j} - \sigma_{,\alpha}(A^{i\alpha}_{.k.l.j} - A^{i\alpha}_{.l.k.j}), \tag{1.6}$$

where a subscript preceded by a comma indicates covariant differentiation with respect to the Γ's.

The conditions of integrability of (1.5) and $\sigma_{,i} = 0$ furnish the complete set of conformal invariants, for the two metrics f and f'. Before outlining the method for obtaining these invariants it may be worth noticing the essential difference between our problem and the corresponding one for a Riemann space. If f is a Riemann metric, the coefficient of $\sigma_{,\alpha}$ in (1.5) vanishes identically and the algebraic elimination of the $\frac{n(n+1)}{2}$ quantities of $\sigma_{,\alpha,\beta}$ carries with it the elimination of $\sigma_{,\alpha}\, \sigma_{,\beta}$ so that the result of elimination is a tensor, the well-known conformal curvature tensor of Weyl. In the generalized metric space the elimination of $\sigma_{,\alpha,\beta}$ gives a conformal invariant which in general is not a tensor while the elimination of the $n^2 + 2n$ quantities $\sigma_{,\alpha,\beta}$, $\sigma_{,\alpha}\, \sigma_{,\beta}$ and $\sigma_{,\alpha}$ if it can be performed, will give a conformal tensor.

2. An invariant which we call a *conformal connection* may be defined in two ways, each of which involves an additional assumption about the metric f. The first connection is analogous to the one developed by J. M. Thomas[2] and the second one by T. Y. Thomas.[3] For Riemann spaces and in fact for generalized metric spaces whose Hessian with respect to dx is a point function these connections are identical, both assumptions being valid for spaces of this sort.

Let f be any generalized metric. With T. Y. Thomas we let $F \equiv f\, |\, g_{\alpha\beta} \,|^{-1/n}$, F being a relative scalar of weight $-2/n$. We assume that

the Hessian of F with respect to dx does not vanish identically. Let $x^i = \varphi^i(\bar{x})$ be any analytic transformation of coördinates whose jacobian $(x,\bar{x}) \not\equiv 0$; then if $G_{ij} = {}^1/_2 F_{.i.j}$,

$$\overline{G}_{ij} = (x,\bar{x})^{-2/n}\, G_{\alpha\beta}\, u_i^\alpha u_j^\beta, \tag{2.1}$$

where $u_i^\alpha = \partial x^\alpha/\partial \bar{x}^i$ and if G^{ij} is the normalized cofactor of G_{ij} in $\mid G_{\alpha\beta} \mid$

$$\overline{G}^{ij} = (x,\, \bar{x})^{2/n}\, G^{\alpha\beta}\, v_\alpha^i v_\beta^j \tag{2.2}$$

where $v_\alpha^i = \partial \bar{x}^i/\partial x^\alpha$.

When we form the affine connection out of the G's (cf. (1.4)) we find that its components K_{jk}^i transform according to the law

$$\overline{K}_{jk}^i u_s^\alpha = K_{\beta\gamma}^\alpha u_j^\beta u_k^\gamma + u_{jk}^\alpha + \overline{A}^{is}_{.j.k}\bar{\psi}_s u_i^\alpha, \tag{2.3}$$

where

$$u_{jk}^\alpha = \partial^2 x^\alpha/\partial\bar{x}j\partial\bar{x}^k \quad , \quad \bar{\psi}_s = 1/n\, \partial \log (x,\bar{x})/\partial\bar{x}^s \tag{2.4}$$

and

$$\overline{A}^{is} = {}^1/_2\, \overline{F}\, \overline{G}^{is} - d\bar{x}^i d\bar{x}^s.$$

The invariant whose components are K_{jk}^i we call the *conformal connection* and if P_{jkl}^i is the analogue of the curvature tensor formed out of the K's, its law of transformation is

$$\overline{P}_{jkl}^i = P_{\beta\gamma\delta}^\alpha u_j^\beta u_k^\gamma u_l^\delta v_\alpha^i + (\overline{A}^{i\alpha}_{.j.k}\bar{\psi}_{/\alpha/l} - \overline{A}^{i\alpha}_{.j.l}\bar{\psi}_{/\alpha/k}) - \bar{\psi}_{/\alpha}\bar{\psi}_{/\beta}\,(\overline{A}^{h\alpha}_{.k}\overline{A}^{i\beta}_{.h.l} - \overline{A}^{h\alpha}_{.l}\overline{A}^{i\beta}_{.h.k})_{.j} + \bar{\psi}_{/\alpha}\,(\overline{A}^{i\alpha}_{.k/l.j} - \overline{A}^{i\alpha}_{.l/k.j}), \tag{2.5}$$

a subscript preceded by a solidus denoting covariant differentiation with respect to the K's. When the expressions for $\overline{P}_{jkl}^i + {}^1/_2\,\overline{P}_{ijk}^i$ are evaluated we get $n(n+1)$ 2 equations that may be solved for the $n(n+1)$ 2 quantities $\bar{\psi}_{/\alpha/\beta}$ giving expressions of the form

$$\bar{\psi}_{/\alpha/\beta} = S_{\alpha\beta}^{jk}[\overline{P}_{jki}^i + {}^1/_2\overline{P}_{ijk}^i - (P_{\lambda\mu i}^i + {}^1/_2 P_{i\lambda\mu}^i)u_j^\lambda u_k^\mu] + T_{\alpha\beta}^{jk}\bar{\psi}_{/j}\bar{\psi}_{/k} + U_{\alpha\beta}^j\bar{\psi}_{/j}. \tag{2.6}$$

By means of (2.6) we are able to eliminate higher derivatives of $\bar{\psi}$ and thus obtain an infinite sequence of conformal invariants which are not tensors in general. These invariants together with $K^i_{jk.l_1.l_2\ldots/m_1/m_2\ldots}$ and $G_{ij.k_1.k_2\ldots}$ constitute a complete set of conformal invariants, the proof of this fact being analogous to that used in the metric equivalence of generalized spaces.

3. By making a further assumption about $f(x, dx)$ we can form a conformal connection which leads to conformal tensors. This assumption is false for spaces whose Hessian is a point function and consequently false for Riemann spaces. That it is not always false can be shown by a simple example.

From (2.3) we obtain by contraction

$$\overline{K}^h_{hk} = K^\alpha_{\alpha\beta}u^\beta_k + \overline{\psi}_s(n\delta^s_k + \overline{A}^{hs}_{.h.k})$$

or

$$\overline{K}^h_{hk} = K^\alpha_{\alpha\beta}u^\beta_k + \overline{\psi}_s\,(^1/_2\overline{F}\,\overline{G}^{hs}_{.h})_{.k}. \tag{3.1}$$

We denote the absolute conformal tensor $(^1/_2FG^{hs}_{.h})_{.k}$ by T^s_k. If we assume that $|\,T^s_k\,| \not\equiv 0$, we may define S^k_m by

$$T^s_k\,S^k_m = \delta^s_m. \tag{3.2}$$

Equations (3.1) may then be solved for $\overline{\psi}_s$ giving

$$\overline{\psi}_s = \overline{K}^h_{hr}\overline{S}^r_s - K^h_{hr}S^r_\alpha u^\alpha_s.$$

Hence if we define an invariant whose components are

$$C^i_{jk} = {}^1/_2(K^i_{\alpha\beta}dx^\alpha dx^\beta - A^{ia}K^h_{h\beta}S^\beta_\alpha)_{.j.k}, \tag{3.3}$$

we shall find their law of transformation to be

$$\overline{C}^i_{jk}u^\alpha_i = C^\alpha_{\beta\gamma}u^\beta_j u^i_k + u^\alpha_{jk}. \tag{3.4}$$

Thus, if this invariant exists the complete set of conformal tensors is obtained by replacing Γ^i_{jk} in the complete set of metric tensors by the corresponding C^i_{jk}.

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[1] L. Berwald, *Jahr. der Deut. Math. Ver.*, 1925, **34**, pp. 213–20.

[2] These PROCEEDINGS, **12**, (1926), 389–393.

[3] *Ibid.*, **12**, (1926), 352–359.